p-adic properties of sequences and finite state automata

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Apéry numbers

- $1, 5, 73, 1445, 33001, 819005, 21460825, \dots$
- These numbers were famously used by Apéry in his unexpected proof of the irrationality of $\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3}$.

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- The Apéry numbers satisfy the recursion

$$A(n+1) = \frac{(2n+1)(an^2 + an + b)A(n) - n(cn^2 + d)A(n-1)}{(n+1)^3},$$

with $(a, b, c, d) = (17, 5, 1, 0)$ and $A(-1) = 0,\ A(0) = 1$.

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- We get integer solutions only for very few other choices of (a, b, c, d). The resulting sequences are called **Apéry-like**.

The Apéry numbers grow very fast, very quickly!

$$\begin{aligned}A(514) = {} & 18302895814171100915047092006619847874140183527500332718489776281989 25\\ & 6185126381909416836091946547570740452866928890650747994105651993258455\\ & 7633911393542031430488526498980743703754634293456985928723284056998909\\ & 9913128982648365723614621605942880743295567135010618701762093782414932\\ & 4069850849365310472593739491145802486900280136902089215111475384509858\\ & 0727023685768554922266793138265201632707069550556257442361953600440506\\ & 5102295575537993999776855645628509479896671562759824334324988255451384\\ & 3266473790293791513427625590011612036536525394613722954096000733290654\\ & 9383802754339120934940473636170233440832465458917665036163012134767347\\ & 4358914151916199364199805165053966151864601189955610708798835455451704\\ & 7098957232120659258014966494724386464808379665263593151922753262347807\\ & 8027172617073\\ \equiv {} & 1 \pmod 8\end{aligned}$$

- Apéry numbers $A(n)$ are the *diagonal Taylor coefficients* of

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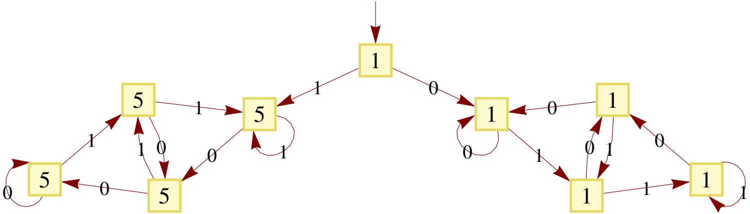
This automatically generated automaton can be simplified!



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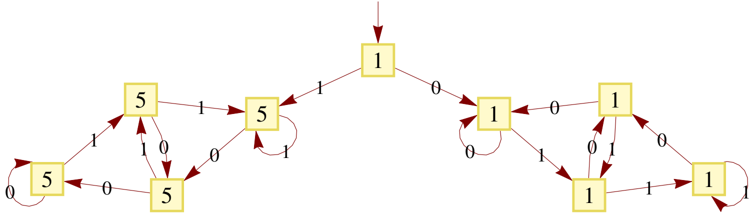

- For instance: $A(514) = A(1000000010_{\text{base }2}) \equiv 1 \pmod 8$.
- Actually, we immediately see that $A(n) \equiv \begin{cases} 1, & \text{if } n \text{ is even,} \\ 5, & \text{if } n \text{ is odd.} \end{cases}$

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Example

Moreover, the Almkvist–Zudilin numbers, defined as

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3},$$

are periodic modulo 8.

$1, 3, 9, 3, -279, -2997, -19431, -65853, 292329, \ldots$

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Theorem (DDMSW, Lucas congruences)

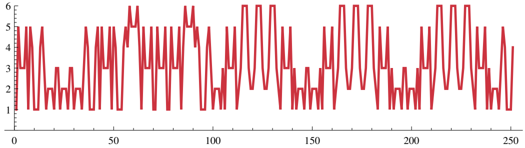
All Apéry-like sequences $C(n)$ satisfy Lucas congruences for all primes p. That is, if $n = n_0 + n_1 p + \ldots + n_r p^r$ is the expansion of n in base p, then

$$C(n) \equiv C(n_0)C(n_1)\ldots C(n_r) \pmod{p}.$$

Example

$$A(514) = A(4024_{\text{base } 5}) \equiv A(4)A(0)A(2)A(4) \equiv 3 \pmod{5}$$

- Values of $A(n)$ modulo 7:



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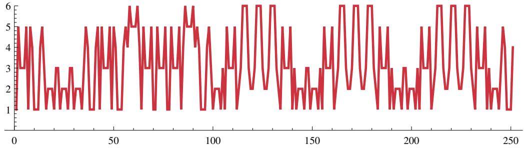

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For any prime p, and $n = 0, 1, \ldots, p-1$, the Apéry numbers $A(n)$ satisfy

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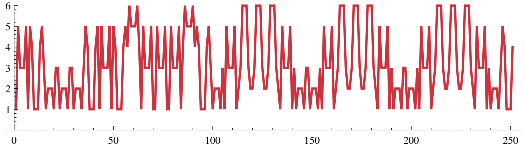

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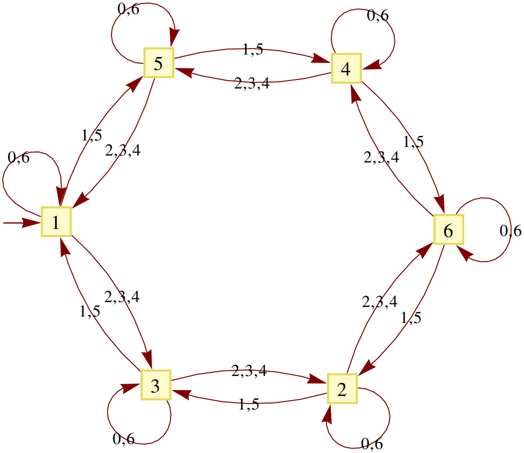
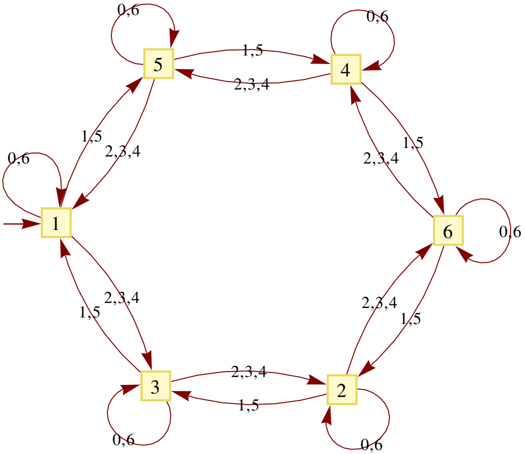
- Also: residue 0 does not occur modulo 7!

Finite state automaton for $A(n) \pmod 7$:



Finite state automaton for $A(n)$ (mod 7):



No vertex for 0.

- Other primes never dividing any Apéry number:

$$2, 3, 7, 13, 23, 29, 43, 47, 53, 67, 71, 79, 83, 89, \ldots$$

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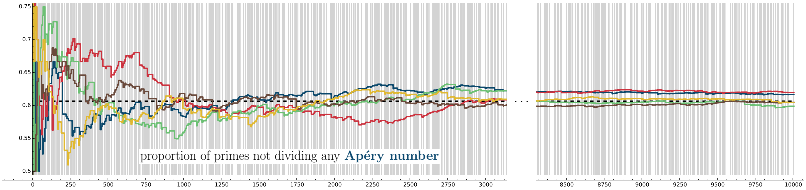

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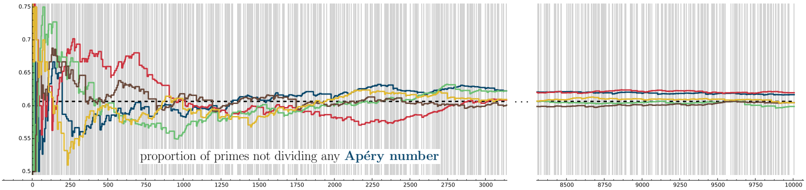

Could it be $e^{-1/2} \approx 60.65\%$? Based on heuristic probabilistic arguments and

- Lucas congruences,
- palindromic behavior of Apéry numbers,
- $e^{-1/2} = \lim_{p \to \infty} \left(1 - \frac{1}{p}\right)^{(p+1)/2}$.

- For one Apéry-like sequence, namely

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right),$$

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Thank you!

Our experiments were fueled by:

- Sage
  open-source, free computer algebra system based on python
- TeXmacs
  open-source, free WYSIWYG TeX-quality editor