p-adic properties of sequences and finite state automata

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Can you identify the following sequence?

$$1, 1, 2, 3, 5, 8,$$

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$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

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Fibonacci numbers

Recursive relation:

$$F_{-1} = 0, \quad F_0 = 1,$$

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 0.$$

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Generating function:
Fibonacci numbers F_n are given by the *Taylor coefficients* of

$$\frac{1}{1 - x - x^2}.$$

Another famous sequence

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

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Apéry numbers

Recursive relation: $(a, b, c, d) = (17, 5, 1, 0)$

$$A_{-1} = 0, \ A_0 = 1,$$
$$A_{n+1} = \frac{(2n+1)(an^2 + an + b)A_n - n(cn^2 + d)A_{n-1}}{(n+1)^3}.$$

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Generating function:
Apéry numbers A_n are given by the *diagonal Taylor coefficients* of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4}.$$

Another famous sequence

Apéry numbers A_n:

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$A_n \pmod 3$:

$$1, 2, 1, 2, 1, 2,$$

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$A_n \pmod 5$:

$$1, 0, 3, 0, 1, 0, 0, 0, 0, 0,$$

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$A_n \pmod 7$:

$$1, 5, 3, 3, 3, 5, 1, 5, 4, 1, 1, 1, 4, \ldots$$

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NO zero here!

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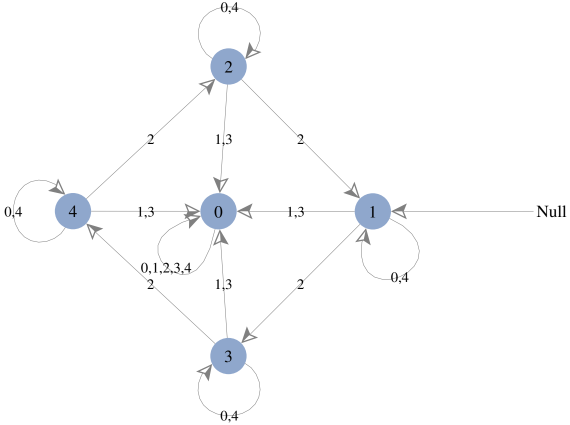
$$1, 5, 3, 3, 3, 5, 1, 5, 4, 1, 1, 1, 4, \ldots$$

NO zero here! One can prove which residues appear by using

finite state automata!

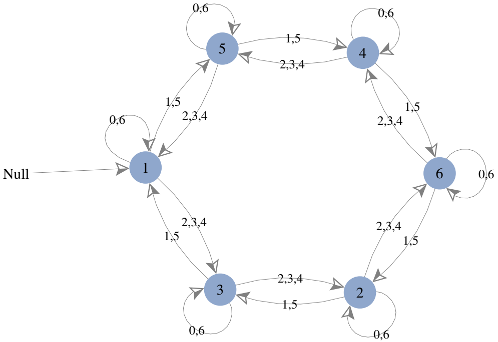
Finite state automaton for $A_n \pmod 5$:



For instance: $514 = 4024_{\text{base } 5} \xrightarrow{\text{FSA}} A(514) \equiv 3 \pmod 5$

$$
\begin{aligned}
A(514) = {} & 1830289581417110091504709200661984787414018352750033271848977628198925\\
& 6185126381909416836091946547570740452866928890650747994105651993258455\\
& 7633911393542031430488526498980743703754634293456985928723284056998909\\
& 9913128982648365723614621605942880743295567135010618701762093782414932\\
& 4069850849365310472593739491145802486900280136902089215111475384509858\\
& 0727023685768554922266793138265201632707069550556257442361953600440506\\
& 5102295575537993999776855645628509479896671562759824334324988255451384\\
& 3266473790293791513427625590011612036536525394613722954096000733290654\\
& 9383802754339120934940473636170233440832465458917665036163012134767347\\
& 4358914151916199364199805165053966151864601189955610708798835455451704\\
& 7098957232120659258014966494724386464808379665263593151922753262347807\\
& 8027172617073\\
\equiv {} & 3 \pmod 5
\end{aligned}
$$

Finite state automaton for $A_n \pmod 7$:



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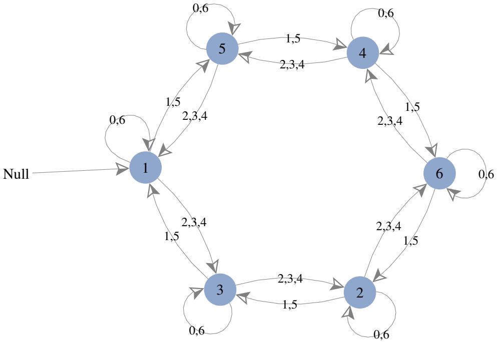

No vertex for 0.

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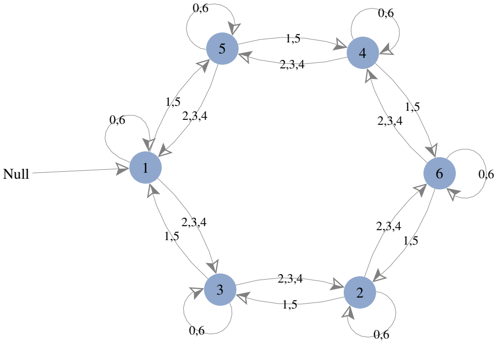

No vertex for 0.

Other primes never dividing any Apéry number:

$$2, 3, 7, 13, 23, 29, 43, 47, 53, 67, 71, 79, 83, 89, \ldots$$

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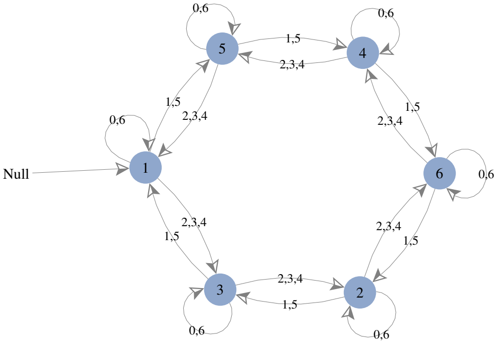

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Experimentally: about 60%.

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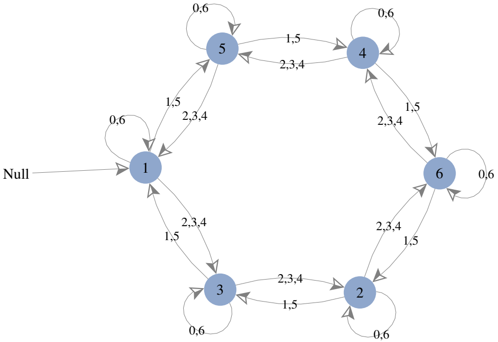

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We are systematically studying all Apéry-like sequences. For instance:

- modulo which primes are they periodic,
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