Math 415 - Final Exam

Friday, December 12, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

To be completed by the grader:

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Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of ? pages. Take a moment to make sure you have all pages.
- You have 180 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Important Note

- The collection of problems below is not representative of the final exam!
- The first three problems cover the material since the third midterm exam, and problems on the final exam on these topics will be of similar nature.
- Problems 4 and 5 are a good start to review the material we covered earlier; however, on the exam itself you should expect questions of the kind that we had on the previous midterms.
- In other words, **to prepare for the final, you need to also prepare our past midterm exams and practice exams**.
- In particular, a basic understanding of Fourier series or the ability to work with spaces of polynomials are expected.

Problem 1. Find a solution to the initial value problem (that is, differential equation plus initial condition)

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \boldsymbol{u}, \quad \boldsymbol{u}(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

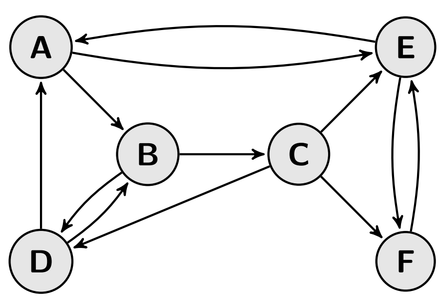
Simplify your solution as far as possible.

Problem 2. The processors of a supercomputer are inspected weekly in order to determine their condition. The condition of a processor can either be perfect, good, reasonable or bad.

A perfect processor is still perfect after one week with probability 0.7, with probability 0.2 the state is good, and with probability 0.1 it is reasonable. A processor in good conditions is still good after one week with probability 0.6, reasonable with probability 0.2, and bad with probability 0.2. A processor in reasonable condition is still reasonable after one week with probability 0.5 and bad with probability 0.5. A bad processor must be repaired. The reparation takes one week, after which the processor is again in perfect condition.

In the steady state, what is percentage of processors in perfect condition?

Problem 3. Determine the PageRank vector for the following system of webpages, and rank the webpages accordingly.



Problem 4. Consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$.
(b) Determine the LU decomposition of A.
(c) Determine the inverse of A.
(d) What is the determinant of A?
(e) Determine the QR decomposition of A.
(f) Determine the eigenvalues of A and find bases for the eigenspaces.
(g) Diagonalize A.

Problem 5. Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) Find orthogonal bases for all four fundamental subspaces.
(b) Determine the projection matrices corresponding to orthogonal projection onto $\mathrm{Col}(A)$ and $\mathrm{Col}(A^T)$.
(c) Consider the linear function $T : \mathbb{R}^2 \to \mathbb{R}^3$, which maps $\boldsymbol{x}$ to $A\boldsymbol{x}$.
   - Determine the matrix which represents T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.
   - Determine the matrix which represents T with respect to the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\mathbb{R}^2$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\mathbb{R}^3$.

(d) Find the least squares solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

MULTIPLE CHOICE

(? questions, 2 points each)

Instructions for multiple choice questions

- No reason needs to be given. There is always exactly one correct answer.
- Enter your answer on the scantron sheet that is included with your exam.
  In addition, on your exam paper, circle the choices you made on the scantron sheet.
- Use a **number 2 pencil** to shade the bubbles completely and darkly.
- Do **NOT** cross out your mistakes, but rather erase them thoroughly before entering another answer.
- Before beginning, please code in your name, UIN, and netid in the appropriate places. In the **'Section' field on the scantron**, please enter

  000 if Armin Straub is your instructor,

  001 if Philipp Hieronymi is your instructor.

The actual exam will have multiple choice questions here.

The midterm exams as well as the practice exams have plenty of problems that you can (and should) look at again. Below are the short problems and multiple choice questions from the conflict exam of our midterms.

Shorts 1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute $A^T A$.

Shorts 2. Let A be a matrix such that, for every $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ x+y \\ 2x+z \end{bmatrix}$.

Then, what is A?

Shorts 3. Let C be a 3×3 matrix such that C has three pivot columns, and let $\boldsymbol{d}$ be a vector in $\mathbb{R}^3$. Is it true that, if the equation $C\boldsymbol{x} = \boldsymbol{d}$ has a solution, then it has infinitely many solutions?

(a) True.
(b) False.
(c) Unable to determine.

Shorts 4. Let

$$A = \begin{bmatrix} a-1 & a \\ a & a-1 \end{bmatrix}.$$

For which choice(s) of a is the matrix A *not* invertible?

Shorts 5. Write down a 3×3-matrix that is not the zero matrix (ie the matrix whose entries are all zero) and is not invertible.

Shorts 6. Let W_1 be the set of all 2×2-matrices A such that A is invertible, and let W_2 be the set of all 2×2-matrices A such that $A^T = -A$. Are these sets subspaces of the vector space of all 2×2-matrices?

(a) Both W_1 and W_2 are subspaces.
(b) Only W_1 is a subspace.
(c) Only W_2 is a subspace.
(d) Neither W_1 nor W_2 are subspaces.

Shorts 7. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Which of the following is true?

(a) W is empty.
(b) W is a line.
(c) W is a plane.
(d) W is all of $\mathbb{R}^3$.

Shorts 8. Let H be a subspace of $\mathbb{R}^6$ with basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4, \boldsymbol{b}_5\}$. What is the dimension of H?

Shorts 9. Which of the following collections of vectors is linearly independent?

(a) $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

(b) $$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) $$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) $$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Shorts 10. Let A be an 4×5 matrix of rank 2. Is it possible to find two linearly independent vectors that are orthogonal to the null space of A? Is it possible to find two linearly independent vectors that are orthogonal to the left null space of A?

(a) Possible for both.
(b) Possible only for the column space.
(c) Possible only for the row space.
(d) Not possible in either case.
(e) Not enough information to decide.

Shorts 11. Let $\mathbb{P}_2$ be the vector space of all polynomials of degree up to 2, and let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(p(t)) = 3p(t) + 2p'(t).$$

Which matrix A represents T with respect to the standard bases?
(Recall that the standard basis for $\mathbb{P}_2$ is given by $1, t, t^2$.)

Shorts 12. Let V be the following subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 - x_2 - 5x_3 = 0, \quad 10x_1 + 2x_2 - 4x_3 = 0 \right\}$$

What is the dimension of V?

Shorts 13. Let a, b be in $\mathbb{R}$. Consider the three vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For which values of a and b are $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ independent?

(a) $a = 0$ and $b = 1$
(b) $a \neq 0$ and $b \neq 1$
(c) $a = 0$ and $b \neq 1$
(d) $a \neq 0$ and $b = 1$

For which values of a and b does $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ have dimension 1?

(a) $a = 0$ and $b = 1$
(b) $a \neq 0$ and $b \neq 1$
(c) $a = 0$ and $b \neq 1$
(d) $a \neq 0$ and $b = 1$

Shorts 14. What is the dimension of the orthogonal complement of

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}?$$

Shorts 15. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix}.$$

(a) What is the dimension of $\operatorname{Nul}(A)$?
(b) What is the dimension of $\operatorname{Col}(A)$?
(c) What is the dimension of $\operatorname{Nul}(A^T)$?
(d) What is the dimension of $\operatorname{Col}(A^T)$?

Shorts 16. Suppose v_1, v_2, v_3, v_4 are four vectors in $\mathbb{R}^3$. Which of the following statements are correct for all such vectors?

(a) Any three of those vectors form a basis of $\mathbb{R}^3$,
(b) these vectors are linearly dependent,
(c) these vectors span $\mathbb{R}^3$,
(d) one of the vectors is a multiple of one of the other vectors.

Shorts 17. Consider the two matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Which of the following is correct?

(a) $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$
(b) $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) \neq \mathrm{Col}(B^T)$
(c) $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$
(d) $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ and $\mathrm{Col}(A^T) \neq \mathrm{Col}(B^T)$

Shorts 18. Let $W = \mathrm{span}\{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}\}$, and let $\boldsymbol{y} = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}$.

Suppose that $\boldsymbol{y} = \boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a}$ is in W and $\boldsymbol{b}$ is orthogonal to W. Then:

(a) $\boldsymbol{a} = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}$

(b) $\boldsymbol{a} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$

(c) $\boldsymbol{a} = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$

(d) none of the above

Shorts 19. Let A be matrix with the property that $A^2 = A$. What is the best you can say about $\det(A)$?

(a) $\det A = 1$
(b) $\det A = \pm 1$
(c) $\det A \neq 0$
(d) $\det A = 1$ or $\det A = 0$

Shorts 20. If A and B are 3×3 matrices with $\det(A) = -2$ and $\det(B) = -1$. What is the determinant of $C = -2B^T BA$?

(a) 4
(b) -8
(c) 8
(d) -16
(e) 16

Shorts 21. Let A, B be two $n \times n$-matrices. Consider the following two statements:

(S1) If $\det(A) = 0$, then two rows or two columns of A are the same, or a row or a column of A is zero.
(S2) If $AB \neq BA$, then $\det(AB) \neq \det(BA)$.

Then:

(a) Statement S1 and Statement S2 are correct.
(b) Only Statement S1 is correct.
(c) Only Statement S2 is correct.
(d) Neither Statement S1 nor Statement S2 is correct.

Shorts 22. Which of the following choices for a makes $\begin{bmatrix} 0 & 0 & 2 \\ 6 & a & 0 \\ 3a & 1 & 0 \end{bmatrix}$ invertible?

(a) any real number except $-\sqrt{2}$ and $\sqrt{2}$
(b) any real number except -2 and 2
(c) just $-\sqrt{2}$ and $\sqrt{2}$
(d) just -2 and 2

Shorts 23. Consider the following two statements:

(T1) If $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ are three orthonormal vectors, then the projection of $\boldsymbol{v_3}$ onto the span of $\boldsymbol{v_1}, \boldsymbol{v_2}$ is $\boldsymbol{v_3}$.
(T2) The Gram–Schmidt process produces from a linearly independent set $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ an orthonormal set $\{\boldsymbol{q_1}, \ldots, \boldsymbol{q_n}\}$ with the property that for each $k \leq n$ the vectors $\{\boldsymbol{q_1}, \ldots, \boldsymbol{q_k}\}$ span the same subspace as $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}\}$.

Then:

(a) Statement T1 and Statement T2 are correct.
(b) Only Statement T1 is correct.
(c) Only Statement T2 is correct.
(d) Neither Statement T1 nor Statement T2 is correct.

Shorts 24. Consider the vector space V of all continuous functions $\mathbb{R} \to \mathbb{R}$, which are periodic with period 2π, together with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(t)g(t)\mathrm{d}t.$$

Let $f(t)$ be in V. Then the orthogonal projection of $f(t)$ onto the span of $\cos(4t)$ is

(a) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} \cos^2(4t)\,\mathrm{d}t}\cos(4t)$

(b) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} \cos^2(4t)\,\mathrm{d}t}f(t)$

(c) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t}f(t)$

(d) $\dfrac{\int_0^{2\pi} f(t)\cos(4t)\,\mathrm{d}t}{\int_0^{2\pi} f(t)^2\mathrm{d}t}\cos(4t)$

(e) none of the above

Shorts 25. Consider the space $\mathbb{P}^3$ of polynomials of degree up to 3, together with the inner product

$$\langle p(t), q(t)\rangle = \int_0^1 p(t)q(t)\mathrm{d}t.$$

Which of the following sets of vectors is orthogonal with respect to this inner product?

(a) $\{1, t\}$
(b) $\{t, t^2\}$
(c) $\{2, -2t\}$
(d) none of the above

Shorts 26. Let A be an $n \times n$ matrix. Consider the following two statements:

(U1) The matrix $8A$ has the same eigenvalues as A.
(U2) The matrix $8A$ has the same eigenvectors as A.

Then:

(a) Statement U1 and Statement U2 are correct.
(b) Only Statement U1 is correct.
(c) Only Statement U2 is correct.
(d) Neither Statement U1 nor Statement U2 is correct.

Shorts 27. Let $W = \operatorname{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ and $\boldsymbol{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. Let $\boldsymbol{w}_1$ be the orthogonal projection of $\boldsymbol{v}_1$ onto W, and let $\boldsymbol{w}_2$ be the orthogonal projection of $\boldsymbol{v}_2$ onto W. Then:

(a) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

(b) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$

(c) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(d) $\boldsymbol{w}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(e) $\boldsymbol{w}_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$