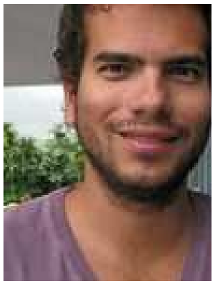
Pre-lecture trivia

Who are these four?

- Artur Avila, Manjul Bhargava, Martin Hairer, Maryam Mirzakhani
- Just won the **Fields Medal**!
  - analog to Nobel prize in mathematics
  - awarded every four years
  - winners have to be younger than 40
  - cash prize: 15,000 C$

Armin Straub
astraub@illinois.edu

Review

- Each linear system corresponds to an augmented matrix.

$$\begin{array}{rrrcr} 2x_1 & -x_2 & & = & 6 \\ -x_1 & +2x_2 & -x_3 & = & -9 \\ & -x_2 & +2x_3 & = & 12 \end{array} \qquad \left[\begin{array}{rrr|r} 2 & -1 & & 6 \\ -1 & 2 & -1 & -9 \\ & -1 & 2 & 12 \end{array}\right]$$

augmented matrix

- To solve a system, we perform **row reduction**.

$$\overset{R2 \to R2 + \frac{1}{2}R1}{\rightsquigarrow} \left[\begin{array}{rrr|r} 2 & -1 & 0 & 6 \\ 0 & \frac{3}{2} & -1 & -6 \\ 0 & -1 & 2 & 12 \end{array}\right]$$

$$\overset{R3 \to R3 + \frac{2}{3}R2}{\rightsquigarrow} \left[\begin{array}{rrr|r} 2 & -1 & 0 & 6 \\ 0 & \frac{3}{2} & -1 & -6 \\ 0 & 0 & \frac{4}{3} & 8 \end{array}\right]$$

echelon form!

- **Echelon form** in general:

$$\left[\begin{array}{ccccccccccc} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading terms in each row are the **pivots**.

Armin Straub
astraub@illinois.edu

Definition 1. A matrix is in **reduced echelon form** if, in addition to being in echelon form, it also satisfies:

- Each pivot is 1.
- Each pivot is the only nonzero entry in its column.

Example 2. Our initial matrix in echelon form put into reduced echelon form:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \blacksquare & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & \blacksquare & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that, to be in reduced echelon form, the pivots $\blacksquare$ also have to be scaled to 1.

Example 3. Are the following matrices in reduced echelon form?

(a) $\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$ YES

(b) $\begin{bmatrix} 1 & 0 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ NO

(c) $\begin{bmatrix} 1 & 0 & -2 & 3 & 2 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ NO

Theorem 4. (Uniqueness of the reduced echelon form) Each matrix is row equivalent to one and only one reduced echelon matrix.

Question. Is the same statement true for the echelon form?

Clearly not; for instance, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are different row equivalent echelon forms.

Armin Straub
astraub@illinois.edu

Example 5. Row reduce to echelon form (often called **Gaussian elimination**) and then to reduced echelon form (often called **Gauss–Jordan elimination**):

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution.

After $R1 \leftrightarrow R3$, we get: ($R1 \leftrightarrow R2$ would be another option; try it!)

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Then, $R2 \to R2 - R1$ yields:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Finally, $R3 \to R3 - \frac{3}{2}R2$ produces the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

To get the reduced echelon form, we first scale all rows:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Then, $R2 \to R2 - R3$ and $R1 \to R1 - 2R3$, gives:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Finally, $R1 \to R1 + 3R2$ produces the reduced echelon form:

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Armin Straub
astraub@illinois.edu

Solution of linear systems via row reduction

After row reduction to echelon form, we can easily solve a linear system.
(especially after reduction to reduced echelon form)

Example 6.

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \rightsquigarrow \begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = & 0 \\ & & x_3 & -8x_4 & & = & 5 \\ & & & & x_5 & = & 7 \end{array}$$

- The pivots are located in columns $1, 3, 5$. The corresponding variables x_1, x_3, x_5 are called **pivot variables** (or **basic variables**).
- The remaining variables x_2, x_4 are called **free variables**.
- We can solve each equation for the pivot variables in terms of the free variables (if any). Here, we get:

$$\begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = & 0 \\ & & x_3 & -8x_4 & & = & 5 \\ & & & & x_5 & = & 7 \end{array} \qquad \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ free} \\ x_5 = 7 \end{cases}$$

- This is the **general solution** of this system. The solution is in parametric form, with parameters given by the free variables.
- Just to make sure: Is the above system consistent? Does it have a unique solution?

Example 7. Find a parametric description of the solution set of:

$$\begin{array}{cccccccc} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = & -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = & 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = & 15 \end{array}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right].$$

We determined earlier that its reduced echelon form is

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

The pivot variables are x_1, x_2, x_5.

The free variables are x_3, x_4.

Hence, we find the general solution as:

$$\begin{cases} x_1 = -24 + 2x_3 - 3x_4 \\ x_2 = -7 + 2x_3 - 2x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \\ x_5 = 4 \end{cases}$$

Related and extra material

- In our textbook: still, parts of 1.1, 1.3, 2.2 (just pages 78 and 79)

  As before, I would suggest waiting a bit before reading through these parts (say, until we covered things like matrix multiplication in class).

- Suggested practice exercise:

  Section 1.3: 13, 20; Section 2.2: 2 (only reduce A, B to echelon form)