Review: matrix multiplication

- $A\boldsymbol{x}$ is a linear combination of the columns of A with weights given by the entries of $\boldsymbol{x}$.

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

- $A\boldsymbol{x} = \boldsymbol{b}$ is the **matrix form** of the linear system with augmented matrix $[\ A \quad \boldsymbol{b}\ ]$.

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \iff \begin{array}{rrcl} 2x_1 & +3x_2 & = & b_1 \\ 3x_1 & +x_2 & = & b_2 \end{array}$$

- Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B.

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 7 & 3 \end{bmatrix}$$

- Matrix multiplication is not commutative: usually, $AB \neq BA$.

A comment on lecture notes

My personal suggestion:

- before lecture: have a quick look (15min or so) at the pre-lecture notes to see where things are going
- during lecture: take a minimal amount of notes (everything on the screens will be in the post-lecture notes) and focus on the ideas
- after lecture: go through the pre-lecture notes again and fill in all the blanks by yourself
- then compare with the post-lecture notes

Since I am writing the pre-lecture notes a week ahead of time, there is usually some minor differences to the post-lecture notes.

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Transpose of a matrix

Definition 1. The **transpose** A^T of a matrix A is the matrix whose columns are formed from the corresponding rows of A. rows $\leftrightarrow$ columns

Example 2.

(a) $\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}$

(b) $[\, x_1 \;\; x_2 \;\; x_3 \,]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

A matrix A is called **symmetric** if $A = A^T$.

Theorem 3. Let A, B be matrices of appropriate size. Then:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$ (illustrated by last practice problems)

Example 4. Deduce that $(ABC)^T = C^T B^T A^T$.

Solution. $(ABC)^T = ((AB)\,C)^T = C^T (AB)^T = C^T B^T A^T$

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Back to matrix multiplication

Review. Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B.

Two more ways to look at matrix multiplication

Example 5. What is the entry $(AB)_{i,j}$ at row i and column j?

The j-th column of AB is the vector $A \cdot (\text{col } j \text{ of } B)$.

Entry i of that is $(\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$. In other words:

$$(AB)_{i,j} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

Use this **row-column rule** to compute:

$$\begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 16 & -3 \\ 0 & 3 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 16 & -3 \\ 0 & 3 \end{bmatrix} \end{array}$$

[Can you see the rule $(AB)^T = B^T A^T$ from here?]

Observe the symmetry between rows and columns in this rule!

It follows that the interpretation

> "Each column of AB is a linear combination of the columns of A with weights given by the corresponding column of B."

has the counterpart

> "Each row of AB is a linear combination of the rows of B with weights given by the corresponding row of A."

Example 6.

(a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 7 & 8 & 9 \end{bmatrix}$

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LU decomposition

Elementary matrices

Example 7.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$

Definition 8. An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

The result of an elementary row operation on A is EA

where E is an elementary matrix (namely, the one obtained by performing the same row operation on the appropriate identity matrix).

Example 9.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

We write $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, but more on inverses soon.

Elementary matrices are **invertible** because elementary row operations are reversible.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

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Practice problems

Example 10. Choose either column or row interpretation to “see” the result of the following products.

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} =$

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