Review

- A **vector space** is a set of vectors which can be added and scaled (without leaving the space!); subject to the "usual" rules.
- The set of all polynomials of degree up to 2 is a vector space.

$$\begin{aligned}[a_0 + a_1 t + a_2 t^2] + [b_0 + b_1 t + b_2 t^2] &= [(a_0 + b_0) + (a_1 + b_1) t + (a_2 + b_2) t^2] \\ r[a_0 + a_1 t + a_2 t^2] &= [(r a_0) + (r a_1) t + (r a_2) t^2]\end{aligned}$$

  Note how it "works" just like $\mathbb{R}^3$.
- The set of all polynomials of degree exactly 2 is not a vector space.

$$\underbrace{[1 + 4t + t^2]}_{\text{degree 2}} + \underbrace{[3 - t - t^2]}_{\text{degree 2}} = \underbrace{[4 + 3t]}_{\text{NOT degree 2}}$$

- An easy test that often works is to check whether the set contains the zero vector. (Works in the previous case.)

Example 1. Let V be the set of all functions $f: \mathbb{R} \to \mathbb{R}$. Is V a vector space?

Solution. Yes!

Addition of functions f and g:

$$(f + g)(x) = f(x) + g(x)$$

Note that, once more, this definition is "component-wise".

Likewise for scalar multiplication.

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Subspaces

Definition 2. A subset W of a vector space V is a **subspace** if W is itself a vector space.

Since the rules like associativity, commutativity and distributivity still hold, we only need to check the following:

$W \subseteq V$ is a subspace of V if

- W contains the zero vector $\mathbf{0}$,
- W is closed under addition, (i.e. if $\boldsymbol{u}, \boldsymbol{v} \in W$ then $\boldsymbol{u} + \boldsymbol{v} \in W$)
- W is closed under scaling. (i.e. if $\boldsymbol{u} \in W$ and $c \in \mathbb{R}$ then $c\boldsymbol{u} \in W$)

Note that "$\mathbf{0}$ in W" (first condition) follows from "W closed under scaling" (third condition). But it is crucial and easy to check, so deserves its own bullet point.

Example 3. Is $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ a subspace of $\mathbb{R}^2$?

Solution. Yes!

- W contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- $\begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix}$ is in W.
- $c\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} ca \\ ca \end{bmatrix}$ is in W.

Example 4. Is $W = \left\{\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \text{ in } \mathbb{R}\right\}$ a subspace of $\mathbb{R}^3$?

Solution. Yes!

- W contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
- $\begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ 0 \\ b_1 + b_2 \end{bmatrix}$ is in W.
- $c\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix}$ is in W.

The subspace W is isomorphic to $\mathbb{R}^2$ (translation: $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix}$) but they are not the same!

Example 5. Is $W = \left\{\begin{bmatrix} a \\ 1 \\ b \end{bmatrix} : a, b \text{ in } \mathbb{R}\right\}$ a subspace of $\mathbb{R}^3$?

Solution. No! Missing $\mathbf{0}$.

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Note: $W = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ is "close" to a vector space.

Geometrically, it is a plane, but it does not contain the origin.

Example 6. Is $W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ a subspace of $\mathbb{R}^2$?

Solution. Yes!

- W contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in W.
- $c \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in W.

Example 7. Is $W = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

Solution. No! W does not contain $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

[If $\mathbf{0}$ is missing, some other things always go wrong as well.

For instance, $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ are not in W.]

Example 8. Is $W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \cup \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

[In other words, W is the set from the previous example plus the zero vector.]

Solution. No! $2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ not in W.

Spans of vectors are subspaces

Review. The **span** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is the set of all their linear combinations. We denote it by $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

In other words, $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all vectors of the form

$$c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \ldots + c_m \boldsymbol{v}_m,$$

where $c_1, c_2, \ldots, c_m$ are scalars.

Theorem 9. If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are in a vector space V, then $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is a subspace of V.

Why?

- $\mathbf{0}$ is in $\operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$
- $[c_1 \boldsymbol{v}_1 + \ldots + c_m \boldsymbol{v}_m] + [d_1 \boldsymbol{v}_1 + \ldots + d_m \boldsymbol{v}_m]$
  $= [(c_1 + d_1) \boldsymbol{v}_1 + \ldots + (c_m + d_m) \boldsymbol{v}_m]$
- $r[c_1 \boldsymbol{v}_1 + \ldots + c_m \boldsymbol{v}_m] = [(r c_1) \boldsymbol{v}_1 + \ldots + (r c_m) \boldsymbol{v}_m]$

Example 10. Is $W = \left\{ \begin{bmatrix} a+3b \\ 2a-b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

Solution. Write vectors in W in the form

$$\begin{bmatrix} a+3b \\ 2a-b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 3b \\ -b \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

to see that

$$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}.$$

By the theorem, W is a vector space. Actually, $W = \mathbb{R}^2$.

Example 11. Is $W = \left\{ \begin{bmatrix} -a & 2b \\ a+b & 3a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ a subspace of $M_{2\times 2}$, the space of 2×2 matrices?

Solution. Write "vectors" in W in the form

$$\begin{bmatrix} -a & 2b \\ a+b & 3a \end{bmatrix} = a \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix} + b \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

to see that

$$W = \operatorname{span}\left\{ \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\}.$$

By the theorem, W is a vector space.

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Practice problems

Example 12. Are the following sets vector spaces?

(a) $W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + 3b = 0, 2a - c = 1 \right\}$

No, W_1 does not contain $\mathbf{0}$.

(b) $W_2 = \left\{ \begin{bmatrix} a + c & -2b \\ b + 3c & c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$

Yes, $W_2 = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right\}$.

Hence, W_2 is a subspace of the vector space $\mathrm{Mat}_{2 \times 2}$ of all 2×2 matrices.

(c) $W_3 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : ab \geqslant 0 \right\}$

No. For instance, $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is not in W_3.

(d) W_4 is the set of all polynomials $p(t)$ such that $p'(2) = 1$.

No. W_4 does not contain the zero polynomial.

(e) W_5 is the set of all polynomials $p(t)$ such that $p'(2) = 0$.

Yes. If $p'(2) = 0$ and $q'(2) = 0$, then $(p + q)'(2) = 0$. Likewise for scaling.

Hence, W_5 is a subspace of the vector space of all polynomials.

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