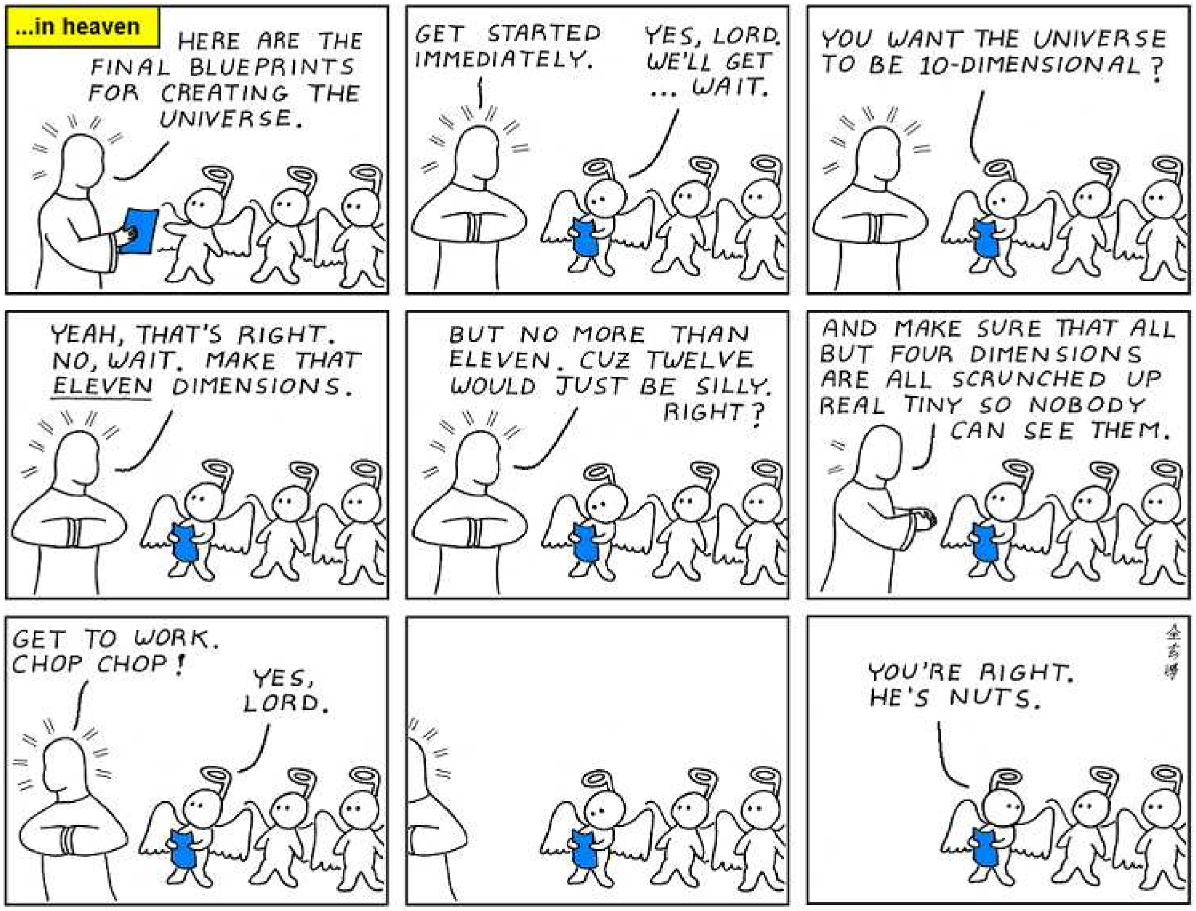
Organizational

- Interested in joining class committee?

  meet ~ 3 times to discuss ideas you may have for improving class

Next: bases, dimension and such



http://abstrusegoose.com/235

Armin Straub
astraub@illinois.edu

Solving $Ax=0$ and $Ax=b$

Column spaces

Definition 1. The **column space** $\mathrm{Col}(A)$ of a matrix A is the span of the columns of A.

If $A = [\ \boldsymbol{a}_1\ \ \ldots\ \ \boldsymbol{a}_n\ ]$, then $\mathrm{Col}(A) = \mathrm{span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$.

- In other words, $\boldsymbol{b}$ is in $\mathrm{Col}(A)$ if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

  Why? Because $A\boldsymbol{x} = x_1\boldsymbol{a}_1 + \ldots + x_n\boldsymbol{a}_n$ is the linear combination of columns of A with coefficients given by $\boldsymbol{x}$.

- If A is $m \times n$, then $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

  Why? Because any span is a space.

Example 2. Find a matrix A such that $W = \mathrm{Col}(A)$ where

$$W = \left\{ \begin{bmatrix} 2x - y \\ 3y \\ 7x + y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}.$$

Solution. Note that

$$\begin{bmatrix} 2x - y \\ 3y \\ 7x + y \end{bmatrix} = x \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} + y \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

Hence,

$$W = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} = \mathrm{Col}\left(\begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 7 & 1 \end{bmatrix} \right).$$

Armin Straub
astraub@illinois.edu

Null spaces

Definition 3. The **null space** of a matrix A is

$$\mathrm{Nul}(A) = \{\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{0}\}.$$

In other words, if A is $m \times n$, then its null space consists of those vectors $\boldsymbol{x} \in \mathbb{R}^n$ which solve the **homogeneous** equation $A\boldsymbol{x} = \boldsymbol{0}$.

Theorem 4. If A is $m \times n$, then $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Proof. We check that $\mathrm{Nul}(A)$ satisfies the conditions of a subspace:

- $\mathrm{Nul}(A)$ contains $\boldsymbol{0}$ because $A\boldsymbol{0} = \boldsymbol{0}$.
- If $A\boldsymbol{x} = \boldsymbol{0}$ and $A\boldsymbol{y} = \boldsymbol{0}$, then $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \boldsymbol{0}$.

  Hence, $\mathrm{Nul}(A)$ is closed under addition.
- If $A\boldsymbol{x} = \boldsymbol{0}$, then $A(c\boldsymbol{x}) = cA\boldsymbol{x} = \boldsymbol{0}$.

  Hence, $\mathrm{Nul}(A)$ is closed under scalar multiplication.

□

Solving $A\boldsymbol{x} = \mathbf{0}$ yields an *explicit description* of $\operatorname{Nul}(A)$.

By that we mean a description as the span of some vectors.

Example 5. Find an explicit description of $\operatorname{Nul}(A)$ where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \overset{R2 \to R2 - 2R1}{\rightsquigarrow} \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \overset{R1 \to \frac{1}{3}R1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \overset{R1 \to R1 - 2R2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix}$$

From the RREF we read off a parametric description of the solutions $\boldsymbol{x}$ to $A\boldsymbol{x} = \mathbf{0}$. Note that x_2, x_4, x_5 are free.

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

In other words,

$$\operatorname{Nul}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note. The number of vectors in the spanning set for $\operatorname{Nul}(A)$ as derived above (which is as small as possible) equals the number of free variables in $A\boldsymbol{x} = \mathbf{0}$.

Another look at solutions to $A\boldsymbol{x} = \boldsymbol{b}$

Theorem 6. Let $\boldsymbol{x}_p$ be a solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$.

Then every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is a solution to the **homogeneous** equation $A\boldsymbol{x} = \boldsymbol{0}$.

- In other words, $\{\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}\} = \boldsymbol{x}_p + \mathrm{Nul}(A)$.
- We often call $\boldsymbol{x}_p$ a **particular solution**.

  The theorem then says that every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is the sum of a fixed chosen particular solution and some solution to $A\boldsymbol{x} = \boldsymbol{0}$.

Proof. Let $\boldsymbol{x}$ be another solution to $A\boldsymbol{x} = \boldsymbol{b}$.

We need to show that $\boldsymbol{x}_n = \boldsymbol{x} - \boldsymbol{x}_p$ is in $\mathrm{Nul}(A)$.

$$A(\boldsymbol{x} - \boldsymbol{x}_p) = A\boldsymbol{x} - A\boldsymbol{x}_p = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}$$

□

Example 7. Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$.

Using the RREF, find a parametric description of the solutions to $A\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 7 & 5 \\ -1 & -3 & 3 & 4 & 5 \end{bmatrix} \overset{\substack{R2 \to R2 - 2R1 \\ R3 \to R3 + R1}}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 6 & 6 & 6 \end{bmatrix}$$

$$\overset{R3 \to R3 - 2R2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R2 \to \frac{1}{3}R2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R1 \to R1 - 3R2}{\rightsquigarrow} \begin{bmatrix} 1 & 3 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is therefore of the form:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 - 3x_2 + x_4 \\ x_2 \\ 1 - x_4 \\ x_4 \end{bmatrix}$$

Armin Straub
astraub@illinois.edu

$$= \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{x}_p} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{elements of } \operatorname{Nul}(A)}$$

We can see nicely how every solution is the sum of a particular solution $\boldsymbol{x}_p$ and solutions to $A\boldsymbol{x} = \mathbf{0}$.

Note. A convenient way to just find a particular solution is to set all free variables to zero (here, $x_2 = 0$ and $x_4 = 0$).

Of course, any other choice for the free variables will result in a particular solution.

For instance, $x_2 = 1$ and $x_4 = 1$ we would get $\boldsymbol{x}_p = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Practice problems

- True or false?
  - The solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$ form a vector space.

    No, with the only exception of $\boldsymbol{b} = \mathbf{0}$.
  - The solutions to the equation $A\boldsymbol{x} = \mathbf{0}$ form a vector space.

    Yes. This is the null space $\operatorname{Nul}(A)$.

Example 8. Is the given set W a vector space?

If possible, express W as the column or null space of some matrix A.

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x = y + 2z \right\}$

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x - 1 = y + 2z \right\}$

(c) $W = \left\{ \begin{bmatrix} x \\ y \\ x + y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$

Example 9. Find an explicit description of $\operatorname{Nul}(A)$ where

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

Armin Straub
astraub@illinois.edu