Review

- Every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is the sum of a fixed chosen particular solution and some solution to $A\boldsymbol{x} = \boldsymbol{0}$.

  For instance, let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$.

  Every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is of the form:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 - 3x_2 + x_4 \\ x_2 \\ 1 - x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{x}_p} + \underbrace{x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{elements of Nul}(A)}$$

- Is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ equal to $\mathbb{R}^3$?

Linear independence

Review.

- $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all linear combinations

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m.$$

- $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is a vector space.

Example 1. Is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ equal to $\mathbb{R}^3$?

Solution. Recall that the span is equal to

$$\left\{ \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \boldsymbol{x} : \boldsymbol{x} \text{ in } \mathbb{R}^3 \right\}.$$

Hence, the span is equal to $\mathbb{R}^3$ if and only if the system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 1 & 2 & 1 & b_2 \\ 1 & 3 & 3 & b_3 \end{array}\right]$$

is consistent for all b_1, b_2, b_3.

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Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 1 & 2 & 1 & b_2 \\ 1 & 3 & 3 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 2 & 4 & b_3 - b_1 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{array}\right]$$

The system is only consistent if $b_3 - 2b_2 + b_1 = 0$.

Hence, the span does not equal all of $\mathbb{R}^3$.

- What went "wrong"? $\quad \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}\right\}$

  Well, the three vectors in the span satisfy

$$\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

- Hence, $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$.

- We are going to say that the three vectors are **linearly dependent** because they satisfy

$$-3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}.$$

Definition 2. Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are said to be **linearly independent** if the equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_p\boldsymbol{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \ldots = x_p = 0$).

Likewise, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are said to be **linearly dependent** if there exist coefficients $x_1, \ldots, x_p$, not all zero, such that

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_p\boldsymbol{v}_p = \mathbf{0}.$$

Example 3.

- Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ independent?
- If possible, find a linear dependence relation among them.

Solution. We need to check whether the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has more than the trivial solution.
In other words, the three vectors are independent if and only if the system

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \boldsymbol{x} = \mathbf{0}$$

has no free variables.
To find out, we reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a column without pivot, we do have a free variable.
Hence, the three vectors are not linearly independent.
To find a linear dependence relation, we solve this system.
Initial steps of Gaussian elimination are as before:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right] \rightsquigarrow \ldots \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free. $x_2 = -2x_3$, and $x_1 = 3x_3$. Hence, for any x_3,

$$3x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since we are only interested in one linear combination, we can set, say, $x_3 = 1$:

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear independence of matrix columns

- Note that a linear dependence relation, such as

$$3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0},$$

can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \mathbf{0}.$$

- Hence, each linear dependence relation among the columns of a matrix A corresponds to a nontrivial solution to $A\boldsymbol{x} = \mathbf{0}$.

Theorem 4. Let A be an $m \times n$ matrix.

The columns of A are linearly independent.
$\iff$ $A\boldsymbol{x} = \mathbf{0}$ has only the solution $\boldsymbol{x} = 0$.
$\iff$ $\mathrm{Nul}(A) = \{\mathbf{0}\}$
$\iff$ A has n pivots. (one in each column)

Example 5. Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ independent?

Solution. Put the vectors in a matrix, and produce an echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

Since each column contains a pivot, the three vectors are independent.

Example 6. (once again, short version)

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ independent?

Solution. Put the vectors in a matrix, and produce an echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last column does not contain a pivot, the three vectors are linearly dependent.

Special cases

- A set of a single nonzero vector $\{\boldsymbol{v}_1\}$ is always linearly independent.

  Why? Because $x_1\boldsymbol{v}_1 = 0$ only for $x_1 = 0$.

- A set of two vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly independent if and only if neither of the vectors is a multiple of the other.

  Why? Because if $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = 0$ with, say, $x_2 \neq 0$, then $\boldsymbol{v}_2 = -\frac{x_1}{x_2}\boldsymbol{v}_1$.

- A set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ containing the zero vector is linearly dependent.

  Why? Because if, say, $\boldsymbol{v}_1 = \mathbf{0}$, then $\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \ldots + 0\boldsymbol{v}_p = \mathbf{0}$.

- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. In other words:

  Any set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.

  Why?

  Let A be the matrix with columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$. This is a $n \times p$ matrix.

  The columns are linearly independent if and only if each column contains a pivot.

  If $p > n$, then the matrix can have at most n pivots.

  Thus not all p columns can contain a pivot.

  In other words, the columns have to be linearly dependent.

Example 7. With the least amount of work possible, decide which of the following sets of vectors are linearly independent.

(a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

Linearly independent, because the two vectors are not multiples of each other.

(b) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

Linearly independent, because it is a single nonzero vector.

(c) columns of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \end{bmatrix}$

Linearly dependent, because these are more than 3 (namely, 4) vectors in $\mathbb{R}^3$.

(d) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Linearly dependent, because the set includes the zero vector.

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