### Review

• Vectors **v**<sub>1</sub>,..., **v**<sub>p</sub> are **linearly dependent** if

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_p\boldsymbol{v}_p = \boldsymbol{0},$$

and not all the coefficients are zero.

• The columns of A are linearly independent

 $\iff$  each column of A contains a pivot.

• Are the vectors  $\begin{bmatrix} 1\\1\\1\\1 \end{bmatrix}$ ,  $\begin{bmatrix} 1\\2\\3\\3 \end{bmatrix}$ ,  $\begin{bmatrix} -1\\1\\3\\3 \end{bmatrix}$  independent?  $\begin{bmatrix} 1&1&-1\\1&2&1\\1&3&3 \end{bmatrix}$   $\rightsquigarrow \begin{bmatrix} 1&1&-1\\0&1&2\\0&2&4 \end{bmatrix}$   $\rightsquigarrow \begin{bmatrix} 1&1&-1\\0&1&2\\0&0&0 \end{bmatrix}$ 

So: no, they are dependent! (Coeff's  $x_3 = 1$ ,  $x_2 = -2$ ,  $x_1 = 3$ )

• Any set of 11 vectors in  $\mathbb{R}^{10}$  is linearly dependent.

## A basis of a vector space

**Definition 1.** A set of vectors  $\{v_1, ..., v_p\}$  in V is a **basis** of V if

- $V = \text{span}\{v_1, ..., v_p\}$ , and
- the vectors v<sub>1</sub>,..., v<sub>p</sub> are linearly independent.

In other words,  $\{v_1, ..., v_p\}$  in V is a basis of V if and only if every vector w in V can be uniquely expressed as  $w = c_1 v_1 + ... + c_p v_p$ .

**Example 2.** Let 
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
,  $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ .

Show that  $\{e_1, e_2, e_3\}$  is a basis of  $\mathbb{R}^3$ .

It is called the standard basis.

### Solution.

- Clearly, span $\{e_1, e_2, e_3\} = \mathbb{R}^3$ .
- $\{e_1, e_2, e_3\}$  are independent, because

$$\left[\begin{array}{rrrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

has a pivot in each column.

**Definition 3.** V is said to have **dimension** p if it has a basis consisting of p vectors.

This definition makes sense because if V has a basis of p vectors, then every basis of V has p vectors. Why? (Think of  $V = \mathbb{R}^3$ .)

A basis of  $\mathbb{R}^3$  cannot have more than 3 vectors, because any set of 4 or more vectors in  $\mathbb{R}^3$  is linearly dependent.

A basis of  $\mathbb{R}^3$  cannot have less than 3 vectors, because 2 vectors span at most a plane (challenge: can you think of an argument that is more "rigorous"?).

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Example 4. \mathbb{R}^3 has dimension 3.
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Indeed, the standard basis  $\begin{bmatrix} 1\\0\\0 \end{bmatrix}$ ,  $\begin{bmatrix} 0\\1\\0 \end{bmatrix}$ ,  $\begin{bmatrix} 0\\0\\1 \end{bmatrix}$  has three elements.

Likewise,  $\mathbb{R}^n$  has dimension n.

**Example 5.** Not all vector spaces have a finite basis. For instance, the vector space of all polynomials has *infinite dimension*.

Its standard basis is  $1, t, t^2, t^3, ...$ This is indeed a basis, because any polynomial can be written as a unique linear combination:  $p(t) = a_0 + a_1 t + ... + a_n t^n$  for some n.

Recall that vectors in V form a **basis** of V if they span V and if they are linearly independent. If we know the dimension of V, we only need to check one of these two conditions:

Theorem	6.	Suppose	that	V	has	dime	ension	d
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- A set of d vectors in V are a basis if they span V.
- A set of *d* vectors in *V* are a basis if they are linearly independent.

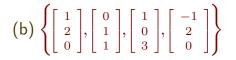
Why?

- If the d vectors were not independent, then d-1 of them would still span V. In the end, we would find a basis of less than d vectors.
- If the d vectors would not span V, then we could add another vector to the set and have d + 1 independent ones.

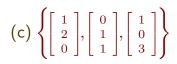
#### **Example 7.** Are the following sets a basis for $\mathbb{R}^3$ ?

 $(\mathsf{a})\left\{ \begin{bmatrix} 1\\2\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1 \end{bmatrix} \right\}$ 

No, the set has less than 3 elements.



No, the set has more than 3 elements.



The set has 3 elements. Hence, it is a basis if and only if the vectors are independent.

		1	0	1		1	0	1 ]
2 1 0	$\sim \rightarrow$	0	1	-2	$\sim \rightarrow$	0	1	-2
0 1 3		0	1	3		0	0	5

Since each column contains a pivot, the three vectors are independent. Hence, this is a basis of  $\mathbb{R}^3$ .

**Example 8.** Let  $\mathbb{P}_2$  be the space of polynomials of degree at most 2.

- What is the dimension of  $\mathbb{P}_2$ ?
- Is  $\{t, 1-t, 1+t-t^2\}$  a basis of  $\mathbb{P}_2$ ?

#### Solution.

• The standard basis for  $\mathbb{P}_2$  is  $\{1, t, t^2\}$ .

This is indeed a basis because every polynomial

$$a_0 + a_1 t + a_2 t^2$$

can clearly be written as a linear combination of  $1, t, t^2$  in a unique way.

Hence,  $\mathbb{P}_2$  has dimension 3.

• The set  $\{t, 1-t, 1+t-t^2\}$  has 3 elements. Hence, it is a basis if and only if the three polynomials are linearly independent.

We need to check whether

$$\underbrace{x_1t + x_2(1-t) + x_3(1+t-t^2)}_{(x_2+x_3) + (x_1-x_2+x_3)t - x_3t^2} = 0$$

has only the trivial solution  $x_1 = x_2 = x_3 = 0$ . We get the equations

$$\begin{aligned}
 x_2 + x_3 &= 0 \\
 x_1 - x_2 + x_3 &= 0 \\
 - x_3 &= 0
 \end{aligned}$$

which clearly only have the trivial solution. (If you don't see it, solve the system!) Hence,  $\{t, 1-t, 1+t-t^2\}$  is a basis of  $\mathbb{P}_2$ .

# Shrinking and expanding sets of vectors

We can find a basis for  $V = \text{span}\{v_1, ..., v_p\}$  by discarding, if necessary, some of the vectors in the spanning set.

**Example 9.** Produce a basis of  $\mathbb{R}^2$  from the vectors

 $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$ 

**Solution.** Three vectors in  $\mathbb{R}^2$  have to be linearly dependent.

Here, we notice that  $\boldsymbol{v}_2 = -2\boldsymbol{v}_1$ .

The remaining vectors  $\{v_1, v_3\}$  are a basis of  $\mathbb{R}^2$ , because the two vectors are clearly independent.

# Checking our understanding

**Example 10.** Subspaces of  $\mathbb{R}^3$  can have dimension 0, 1, 2, 3.

- The only 0-dimensional subspace is **{0**}.
- A 1-dimensional subspace is of the form  $\operatorname{span}\{v\}$  where  $v \neq 0$ .

These subspaces are lines through the origin.

• A 2-dimensional subspace is of the form  $\operatorname{span}\{v, w\}$  where v and w are not multiples of each other.

These subspaces are planes through the origin.

• The only 3-dimensional subspace is  $\mathbb{R}^3$  itself.

True or false?

• Suppose that V has dimension n. Then any set in V containing more than n vectors must be linearly dependent.

That's correct.

- The space P<sub>n</sub> of polynomials of degree at most n has dimension n+1.
   True, as well. A basis is {1, t, t<sup>2</sup>, ..., t<sup>n</sup>}.
- The vector space of functions  $f: \mathbb{R} \to \mathbb{R}$  is infinite-dimensional.

Yes. A still-infinite-dimensional subspace are the polynomials.

• Consider  $V = \text{span}\{v_1, ..., v_p\}$ . If one of the vectors, say  $v_k$ , in the spanning set is a linear combination of the remaining ones, then the remaining vectors still span V.

True,  $\boldsymbol{v}_k$  is not adding anything new.