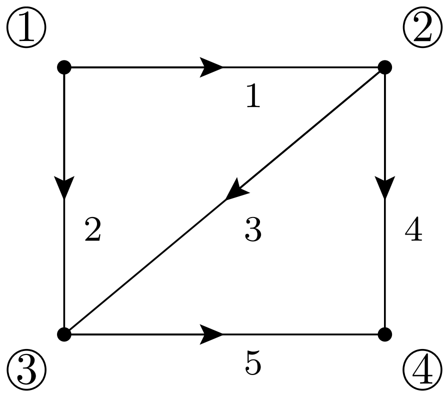
Review

- A graph G can be encoded by the **edge-node incidence matrix**:



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

  - each column represents a node
  - each row represents an edge

- If G has m edges and n nodes, then A is the $m \times n$ matrix with

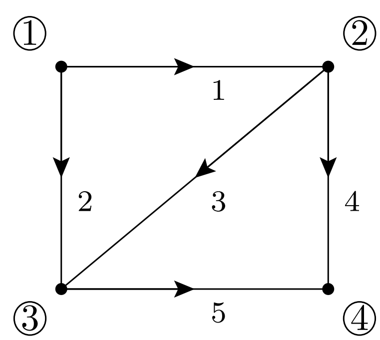
$$A_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j, \\ +1, & \text{if edge } i \text{ enters node } j, \\ 0, & \text{otherwise.} \end{cases}$$

Meaning of the null space

The $\boldsymbol{x}$ in $A\boldsymbol{x}$ is assigning values to each node.

You may think of assigning **potentials** to each node.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ -x_2 + x_3 \\ -x_2 + x_4 \\ -x_3 + x_4 \end{bmatrix}$$



So: $A\boldsymbol{x} = \boldsymbol{0}$

$\iff$ nodes connected by an edge are assigned the same value

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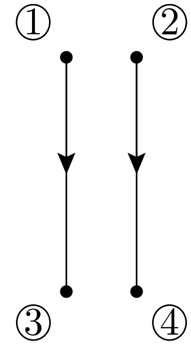
For our graph: $\mathrm{Nul}(A)$ has basis $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

This always happens as long as the graph is **connected**.



Example 1. Give a basis for $\mathrm{Nul}(A)$ for the following graph.

Solution. If $A\boldsymbol{x} = \mathbf{0}$ then $x_1 = x_3$ (connected by edge) and $x_2 = x_4$ (connected by edge).

$\mathrm{Nul}(A)$ has the basis: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Just to make sure: the edge-node incidence matrix is:

$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

> In general:
>
> $\dim \mathrm{Nul}(A)$ is the number of connected subgraphs.

For large graphs, disconnection may not be apparent visually.

But we can always find out by computing $\dim \mathrm{Nul}(A)$ using Gaussian elimination!

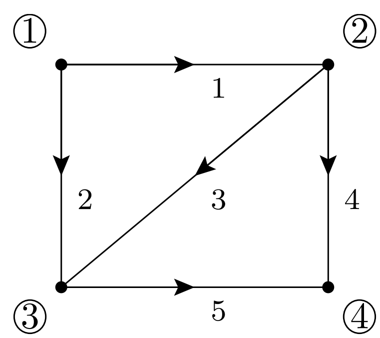
Meaning of the left null space

The $\boldsymbol{y}$ in $\boldsymbol{y}^T A$ is assigning values to each edge.

You may think of assigning **currents** to each edge.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -y_1 - y_2 \\ y_1 - y_3 - y_4 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{bmatrix}$$



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So: $A^T \boldsymbol{y} = \mathbf{0}$

$\Longleftrightarrow$ at each node, (directed) values assigned to edges add to zero

When thinking of currents, this is **Kirchhoff's first law**.
(at each node, incoming and outgoing currents balance)

What is the simplest way to balance current?

Assign the current in a **loop**!

Here, we have two loops: $\text{edge}_1, \text{edge}_3, -\text{edge}_2$ and $\text{edge}_3, \text{edge}_5, -\text{edge}_4$.

Correspondingly, $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ are in $\text{Nul}(A^T)$. Check!

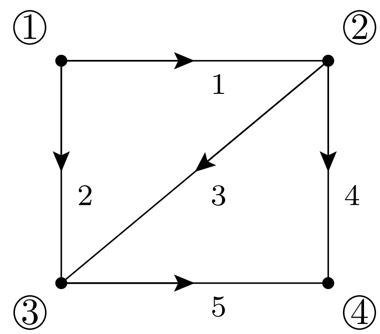
Example 2. Suppose we did not "see" this.

Let us solve $A^T \boldsymbol{y} = \mathbf{0}$ for our graph:

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The parametric solution is $\begin{bmatrix} y_3 - y_5 \\ -y_3 + y_5 \\ y_3 \\ -y_5 \\ y_5 \end{bmatrix}$.

So, a basis for $\text{Nul}(A^T)$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.



Observe that these two basis vectors correspond to loops.

Note that we get the "simpler" loop $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ as $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

In general:
$\dim \text{Nul}(A^T)$ is the number of (independent) loops.

For large graphs, we now have a nice way to computationally find all loops.

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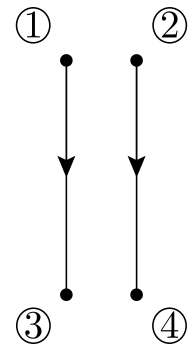
Practice problems

Example 3. Give a basis for $\mathrm{Nul}(A^T)$ for the following graph.



Example 4. Consider the graph with edge-node incidence matrix

$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$.

(a) Draw the corresponding directed graph with numbered edges and nodes.

(b) Give a basis for $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^T)$ using properties of the graph.

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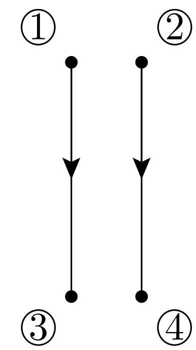
Solutions to practice problems

Example 5. Give a basis for $\mathrm{Nul}(A^T)$ for the following graph.



Solution. This graph contains no loops,

so $\mathrm{Nul}(A^T) = \{\mathbf{0}\}$.

$\mathrm{Nul}(A^T)$ has the empty set as basis (no basis vectors needed).

For comparison: the edge-node incidence matrix

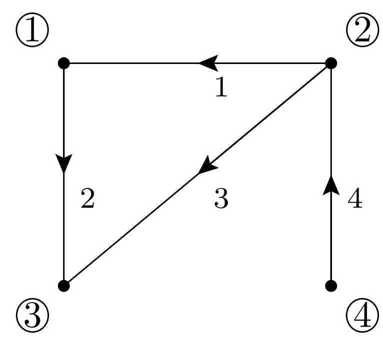
$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

indeed has $\mathrm{Nul}(A^T) = \{\mathbf{0}\}$.

Example 6. Consider the graph with edge-node incidence matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Give a basis for $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^T)$.



Solution.

If $A\boldsymbol{x} = \mathbf{0}$, then $x_1 = x_2 = x_3 = x_4$ (all connected by edges).

$\mathrm{Nul}(A)$ has the basis: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

The graph is connected, so only 1 connected subgraph and $\dim \mathrm{Nul}(A) = 1$.

The graph has one loop: $\mathrm{edge}_1, \mathrm{edge}_2, -\mathrm{edge}_3$

Assign values $y_1 = 1$, $y_2 = 1$, $y_3 = -1$ along the edges of that loop.

$\mathrm{Nul}(A^T)$ has the basis: $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$.

The graph has 1 loop, so $\dim \mathrm{Nul}(A^T) = 1$.

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