Review for Midterm 2

- As of yet unconfirmed:
  - final exam on Friday, December 12, 7–10pm
  - conflict exam on Monday, December 15, 7–10pm

Directed graphs

- Go from directed graph to edge-node incidence matrix A and vice versa.
- Basis for $\mathrm{Nul}(A)$ from connected subgraphs.

  For each connected subgraph, get a basis vector $\boldsymbol{x}$ that assigns 1 to all nodes in that subgraph, and 0 to all other nodes.

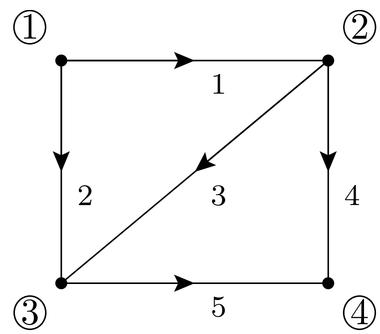
- Basis for $\mathrm{Nul}(A^T)$ from (independent) loops.

  For each (independent) loop, get a basis vector $\boldsymbol{y}$ that assigns 1 and -1 (depending on direction) to the edges in that loop, and 0 to all other edges.

Example 1.

Basis for $\mathrm{Nul}(A)$: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Basis for $\mathrm{Nul}(A^T)$: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$



Fundamental notions

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are **independent** if the only linear relation

  $$c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0}$$

  is the one with $c_1 = c_2 = \ldots = c_n = 0$.

  How to check for independence?

  The columns of a matrix A are independent $\Longleftrightarrow$ $\mathrm{Nul}(A) = \{\boldsymbol{0}\}$.

- Vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ in V are a **basis** for V if
  - they span V, that is $V = \mathrm{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, and
  - they are independent.

  In that case, V has **dimension** n.

- Vectors $\boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{R}^m$ are **orthogonal** if $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + \ldots + v_m w_m = 0$.

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Subspaces

- From an echelon form of A, we get bases for:
  - $\operatorname{Nul}(A)$ — by solving $A\boldsymbol{x} = \boldsymbol{0}$
  - $\operatorname{Col}(A)$ — by taking the pivot columns of A
  - $\operatorname{Col}(A^T)$ — by taking the nonzero rows of the echelon form

  Example 2.

  $$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  Basis for $\operatorname{Col}(A)$: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}$

  Basis for $\operatorname{Col}(A^T)$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$

  Basis for $\operatorname{Nul}(A)$: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$

  Dimension of $\operatorname{Nul}(A^T)$: 2

- The solutions to $A\boldsymbol{x} = \boldsymbol{b}$ are given by $\boldsymbol{x}_p + \operatorname{Nul}(A)$.
- The **fundamental theorem** states that
  - $\operatorname{Nul}(A)$ and $\operatorname{Col}(A^T)$ are orthogonal complements

    So: $\dim \operatorname{Nul}(A) + \dim \operatorname{Col}(A^T) = n$ (number of columns of A)
  - $\operatorname{Nul}(A^T)$ and $\operatorname{Col}(A)$ are orthogonal complements

    So: $\dim \operatorname{Nul}(A^T) + \dim \operatorname{Col}(A) = m$ (number of rows of A)
  - In particular, if $r = \operatorname{rank}(A)$ (nr of pivots):
    - $\dim \operatorname{Col}(A) = r$
    - $\dim \operatorname{Col}(A^T) = r$
    - $\dim \operatorname{Nul}(A) = n - r$
    - $\dim \operatorname{Nul}(A^T) = m - r$

Example 3. Consider the following subspaces of $\mathbb{R}^4$:

(a) $V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + 2b = 0, \ a + b + d = 0 \right\}$

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(b) $V = \left\{ \begin{bmatrix} a+b-c \\ b \\ 2a+3c \\ c \end{bmatrix} : \ a, b, c \text{ in } \mathbb{R} \right\}$

In each case, give a basis for V and its orthogonal complement.

Try to immediately get an idea what the dimensions are going to be!

Solution.

- First step: express these subspaces as one of the four subspaces of a matrix.

  (a) $V = \mathrm{Nul}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \right)$

  (b) $V = \mathrm{Col}\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \right)$

- Give a basis for each.

  (a) row reductions: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

  basis for V: $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

  (b) row reductions: $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

  (no need to continue; we already see that the columns are independent)

  basis for V: $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

- Use the fundamental theorem to find bases for the orthogonal complements.

  (a) $V^\perp = \mathrm{Col}\left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^T \right)$

  note the two rows are clearly independent.

  basis for $V^\perp$: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

  (b) $V^\perp = \mathrm{Nul}\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}^T \right) = \mathrm{Nul}\left(\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \end{bmatrix} \right)$

  row reductions: $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 3 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2/5 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & 1/5 \end{bmatrix}$

  basis for $V^\perp$: $\begin{bmatrix} 2/5 \\ -2/5 \\ -1/5 \\ 1 \end{bmatrix}$

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Example 4. What does it mean for $Ax=b$ if $\mathrm{Nul}(A)=\{\mathbf{0}\}$?

Solution. It means that if there is a solution, then it is unique.

That's because all solutions to $Ax=b$ are given by $x_p+\mathrm{Nul}(A)$.

Linear transformations

Example 5. Let $T:\mathbb{R}^2\to\mathbb{R}^3$ be the linear map represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}$$

with respect to the bases $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ of $\mathbb{R}^2$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ of $\mathbb{R}^3$.

(a) What is $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$?

(b) Which matrix represents T with respect to the standard bases?

Solution.

The matrix tells us that:

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence, $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = 2\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 11 \end{bmatrix}$.

(b) Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence, $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$.

We already know that $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

So, T is represented by $\begin{bmatrix} 4 & 3 \\ 4 & 4 \\ 6 & 5 \end{bmatrix}$ with respect to the standard bases.

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Check your understanding

Think about why each of these statements is true!

- $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ if and only if $\boldsymbol{b}$ is in $\mathrm{Col}(A)$.

  That's because $A\boldsymbol{x}$ are linear combinations of the columns of A.

- A and A^T have the same rank.

  Recall that the rank of A (number of pivots of A) equals $\dim \mathrm{Col}(A)$.

  So this is another way of saying that $\dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T)$.

- The columns of an $n \times n$ matrix are independent if and only if the rows are.

  Let r be the rank of A, and let A be $m \times n$ for now.

  The columns are independent $\Longleftrightarrow$ $r = n$ (so that $\dim \mathrm{Nul}(A) = 0$).

  But also: the rows are independent $\Longleftrightarrow$ $r = m$.

  In the case $m = n$, these two conditions are equivalent.

- $A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ if and only if $\boldsymbol{b}$ is orthogonal to $\mathrm{Nul}(A^T)$.

  This follows from "$A\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ if and only if $\boldsymbol{b}$ is in $\mathrm{Col}(A)$"
  together with the fundamental theorem, which says that $\mathrm{Col}(A)$ is the orthogonal complement of $\mathrm{Nul}(A^T)$.

- The rows of A are independent if and only if $\mathrm{Nul}(A^T) = \{\mathbf{0}\}$.

  Recall that elements of $\mathrm{Nul}(A)$ correspond to linear relations between the columns of A.

  Likewise, elements of $\mathrm{Nul}(A^T)$ correspond to linear relations between the rows of A.