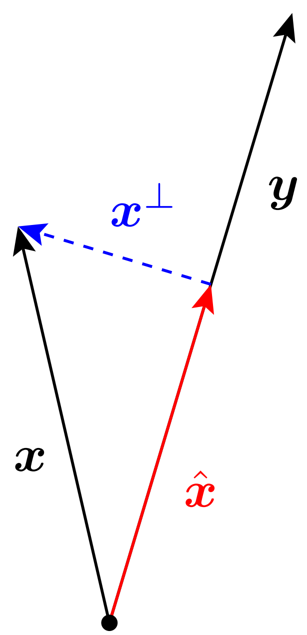
Comments on midterm

Suppose V is a vector space, and you are asked to give a basis.

- CORRECT: V has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
- CORRECT: V has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- OK: $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

  (but you really should point out that the two vectors are independent)
- INCORRECT: $V = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
- INCORRECT: $\text{basis} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Review



- **Orthogonal projection** of $\boldsymbol{x}$ onto $\boldsymbol{y}$:

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}} \boldsymbol{y}.$$

  "Error" $\boldsymbol{x}^\perp = \boldsymbol{x} - \hat{\boldsymbol{x}}$ is orthogonal to $\boldsymbol{y}$.

- If $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ is an **orthogonal basis** of V, and $\boldsymbol{x}$ is in V, then

$$\boldsymbol{x} = c_1 \boldsymbol{y}_1 + \ldots + c_n \boldsymbol{y}_n \qquad \text{with} \quad c_j = \frac{\boldsymbol{x} \cdot \boldsymbol{y}_j}{\boldsymbol{y}_j \cdot \boldsymbol{y}_j}.$$

> $\boldsymbol{x}$ decomposes as the sum of its projections onto each vector in the orthogonal basis.

Example 1. Express $\underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{\boldsymbol{x}}$ in terms of the basis $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_3}$.

Solution. Note that $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ is an orthogonal basis of $\mathbb{R}^3$.

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$$\begin{bmatrix}2\\1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

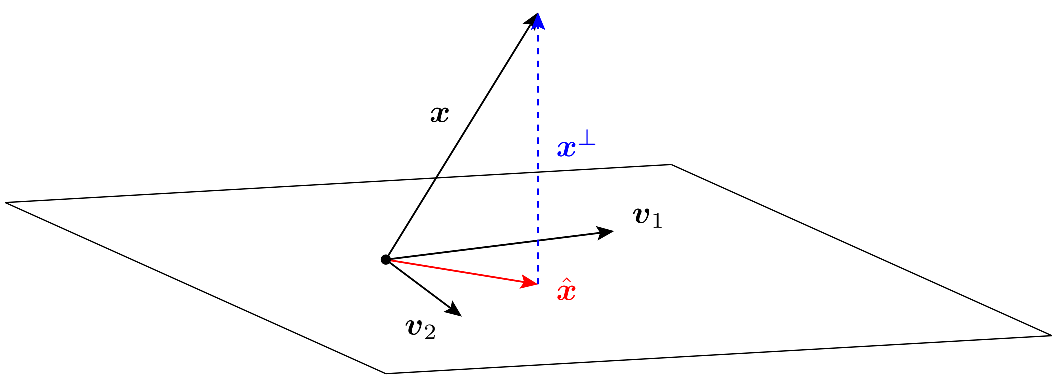
$$= \underbrace{\frac{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{y}_1} + \underbrace{\frac{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{y}_2} + \underbrace{\frac{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}}\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{projection of } \boldsymbol{x} \text{ onto } \boldsymbol{y}_3}$$

$$= \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} + \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Orthogonal projection on subspaces

Theorem 2. Let W be a subspace of $\mathbb{R}^n$. Then, each $\boldsymbol{x}$ in $\mathbb{R}^n$ can be uniquely written as

$$\boldsymbol{x} = \underbrace{\hat{\boldsymbol{x}}}_{\text{in } W} + \underbrace{\boldsymbol{x}^\perp}_{\text{in } W^\perp}.$$



- $\hat{\boldsymbol{x}}$ is the **orthogonal projection** of $\boldsymbol{x}$ onto W.
  $\hat{\boldsymbol{x}}$ is the point in W closest to $\boldsymbol{x}$. For any other $\boldsymbol{y}$ in W, $\operatorname{dist}(\boldsymbol{x}, \hat{\boldsymbol{x}}) < \operatorname{dist}(\boldsymbol{x}, \boldsymbol{y})$.
- If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ is an orthogonal basis of W, then

$$\hat{\boldsymbol{x}} = \left(\frac{\boldsymbol{x}\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\right)\boldsymbol{v}_1 + \ldots + \left(\frac{\boldsymbol{x}\cdot\boldsymbol{v}_m}{\boldsymbol{v}_m\cdot\boldsymbol{v}_m}\right)\boldsymbol{v}_m.$$

  Once $\hat{\boldsymbol{x}}$ is determined, $\boldsymbol{x}^\perp = \boldsymbol{x} - \hat{\boldsymbol{x}}$.
  (This is also the orthogonal projection of $\boldsymbol{x}$ onto $W^\perp$.)

Example 3. Let $W = \operatorname{span}\left\{\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$, and $\boldsymbol{x} = \begin{bmatrix}0\\3\\10\end{bmatrix}$.

- Find the orthogonal projection of $\boldsymbol{x}$ onto W.
  (or: find the vector in W which is closest to $\boldsymbol{x}$)
- Write $\boldsymbol{x}$ as a vector in W plus a vector orthogonal to W.

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Solution.

Note that $\boldsymbol{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are an orthogonal basis for W.

[We will soon learn how to construct orthogonal bases ourselves.]

Hence, the orthogonal projection of $\boldsymbol{x}$ onto W is:

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x} \cdot \boldsymbol{w}_1}{\boldsymbol{w}_1 \cdot \boldsymbol{w}_1} \boldsymbol{w}_1 + \frac{\boldsymbol{x} \cdot \boldsymbol{w}_2}{\boldsymbol{w}_2 \cdot \boldsymbol{w}_2} \boldsymbol{w}_2 = \frac{\begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{10}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$\hat{\boldsymbol{x}}$ is the vector in W which best approximates $\boldsymbol{x}$.

Orthogonal projection of $\boldsymbol{x}$ onto the orthogonal complement of W:

$\boldsymbol{x}^{\perp} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}$. Hence, $\boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}}_{\text{in } W} + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}}_{\text{in } W^{\perp}}$.

Note: Indeed, $\begin{bmatrix} -3 \\ 0 \\ 9 \end{bmatrix}$ is orthogonal to $\boldsymbol{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Definition 4. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be an orthogonal basis of W, a subspace of $\mathbb{R}^n$. Note that the projection map $\pi_W : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\boldsymbol{x} \mapsto \hat{\boldsymbol{x}} = \left(\frac{\boldsymbol{x} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \right) \boldsymbol{v}_1 + \ldots + \left(\frac{\boldsymbol{x} \cdot \boldsymbol{v}_m}{\boldsymbol{v}_m \cdot \boldsymbol{v}_m} \right) \boldsymbol{v}_m$$

is linear. The matrix P representing π_W with respect to the standard basis is the corresponding **projection matrix**.

Example 5. Find the projection matrix P which corresponds to orthogonal projection onto $W = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

Solution. Standard basis : $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

The first column of P encodes the projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$:

$\dfrac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \dfrac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \dfrac{3}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Hence $P = \begin{bmatrix} \frac{9}{10} & * & * \\ 0 & * & * \\ \frac{3}{10} & * & * \end{bmatrix}$.

The second column of P encodes the projection of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$:

$$\frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \text{ Hence } P = \begin{bmatrix} \frac{9}{10} & 0 & * \\ 0 & 1 & * \\ \frac{3}{10} & 0 & * \end{bmatrix}.$$

The third column of P encodes the projection of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$:

$$\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}. \text{ Hence } P = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix}.$$

Example 6. (again)

Find the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$ onto $W = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Solution. $\hat{\boldsymbol{x}} = P\boldsymbol{x} = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$, as in the previous example.

Example 7. Compute P^2 for the projection matrix we just found. Explain!

Solution.

$$\begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{9}{10} & 0 & \frac{3}{10} \\ 0 & 1 & 0 \\ \frac{3}{10} & 0 & \frac{1}{10} \end{bmatrix}$$

Projecting a second time does not change anything anymore.

Practice problems

Example 8. Find the closest point to $\boldsymbol{x}$ in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, where

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. This is the orthogonal projection of $\boldsymbol{x}$ onto $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

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