Review

- The **determinant** is characterized by $\det I = 1$ and the effect of row op's:
  - replacement: does not change the determinant
  - interchange: reverses the sign of the determinant
  - scaling row by s: multiplies the determinant by s
- $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 2 \cdot 3 = 12$
- $\det(A) = 0 \iff A$ is not invertible
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(A^T) = \det(A)$
- What's **wrong**?!

  $\det(A^{-1}) = \det \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}(da - (-b)(-c)) = 1$

  The corrected calculation is:

  $\det \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad - bc)^2}(da - (-b)(-c)) = \frac{1}{ad - bc}$

  This is compatible with $\det(A^{-1}) = \frac{1}{\det(A)}$.

Example 1. Suppose A is a 3×3 matrix with $\det(A) = 5$. What is $\det(2A)$?

Solution. A has three rows.

Multiplying all 3 of them by 2 produces $2A$.

Hence, $\det(2A) = 2^3 \det(A) = 40$.

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A "bad" way to compute determinants

Example 2. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the second column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = -2 \cdot \begin{vmatrix} \blacksquare & - & \blacksquare \\ 3 & \blacksquare & 2 \\ 2 & \blacksquare & 1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & \blacksquare & 0 \\ \blacksquare & + & \blacksquare \\ 2 & \blacksquare & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & \blacksquare & 0 \\ 3 & \blacksquare & 2 \\ \blacksquare & - & \blacksquare \end{vmatrix}$$

$$= -2 \cdot (-1) + (-1) \cdot 1 - 0 = 1$$

Solution. We expand by the third column:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} \blacksquare & \blacksquare & + \\ 3 & -1 & \blacksquare \\ 2 & 0 & \blacksquare \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 & \blacksquare \\ \blacksquare & \blacksquare & - \\ 2 & 0 & \blacksquare \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 & \blacksquare \\ 3 & -1 & \blacksquare \\ \blacksquare & \blacksquare & + \end{vmatrix}$$

$$= 0 - 2 \cdot (-4) + 1 \cdot (-7) = 1$$

Why is the method of cofactor expansion not practical?

Because to compute a large $n \times n$ determinant,

- one reduces to n determinants of size $(n-1) \times (n-1)$,
- then $n(n-1)$ determinants of size $(n-2) \times (n-2)$,
- and so on.

In the end, we have $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ many numbers to add.

WAY TOO MUCH WORK! Already $25! = 15511210043330985984000000 \approx 1.55 \cdot 10^{25}$.

Context: today's fastest computer, Tianhe-2, runs at 34 pflops ($3.4 \cdot 10^{16}$ op's per second).

By the way: "fastest" is measured by computed LU decompositions!

Example 3.

First off, say hello to a new friend: i, the **imaginary unit**

It is infamous for $\boxed{i^2 = -1}$.

$$\begin{vmatrix} 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 1 - i^2 = 2$$

$$\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} - i \begin{vmatrix} i & 0 \\ i & 1 \end{vmatrix} = 2 - i^2 = 3$$

$$\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} - i \begin{vmatrix} i & 0 & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 3 - i^2 \begin{vmatrix} 1 & i \\ i & 1 \end{vmatrix} = 5$$

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$$\begin{vmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & i & \\ & & i & 1 & i \\ & & & i & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & i & & \\ i & 1 & i & \\ & i & 1 & i \\ & & i & 1 \end{vmatrix} - i^2\begin{vmatrix} 1 & i & \\ i & 1 & i \\ & i & 1 \end{vmatrix} = 5 + 3 = 8$$

The Fibonacci numbers!

Do you know about the connection of Fibonacci numbers and rabbits?

Eigenvectors and eigenvalues

Throughout, A will be an $n \times n$ matrix.

Definition 4. An **eigenvector** of A is a nonzero $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{for some scalar } \lambda.$$

The scalar λ is the corresponding **eigenvalue**.

In words: eigenvectors are those $\boldsymbol{x}$, for which $A\boldsymbol{x}$ is parallel to $\boldsymbol{x}$.

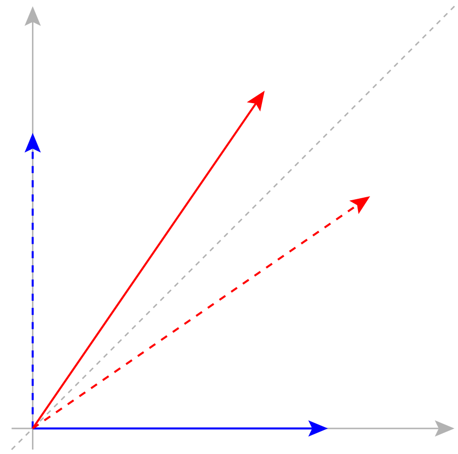
Example 5. Verify that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Solution.

$A\boldsymbol{x} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} = 4\boldsymbol{x}$

Hence, $\boldsymbol{x}$ is an eigenvector of A with eigenvalue 4.

Example 6. Use your geometric understanding to find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



Solution. $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

i.e. multiplication with A is reflection through the line $y = x$.

- $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

  So: $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.

- $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

  So: $\boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = -1$.

Practice problems

Problem 1. Let A be an $n \times n$ matrix.

Express the following in terms of $\det(A)$:

- $\det(A^2) =$
- $\det(2A) =$

  Hint: (unless $n = 1$) this is not just $2\det(A)$

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