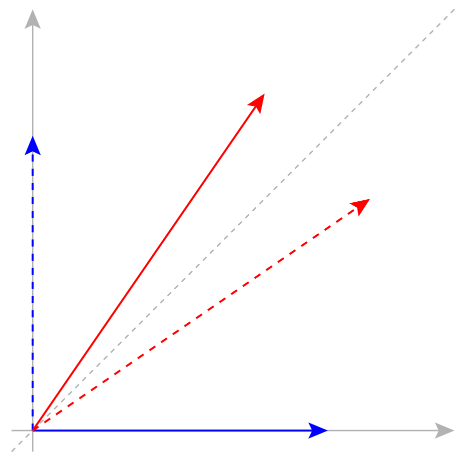
Review

- If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, then $\boldsymbol{x}$ is an **eigenvector** of A with **eigenvalue** λ.
- EG: $\boldsymbol{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$ with eigenvalue 4

  because $A\boldsymbol{x} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} = 4\boldsymbol{x}$
- Multiplication with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is reflection through the line $y = x$.

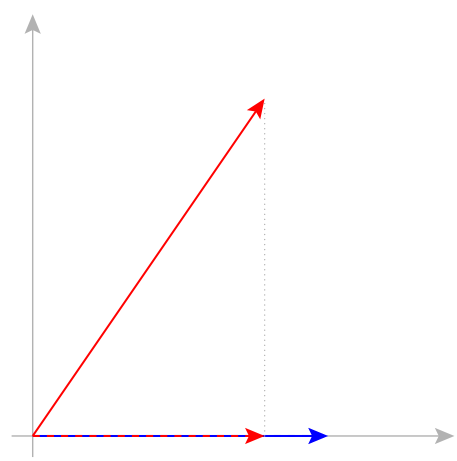


  - $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

    So: $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.
  - $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

    So: $\boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = -1$.

Example 1. Use your geometric understanding to find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.



Solution. $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

i.e. multiplication with A is projection onto the x-axis.

- $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

  So: $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.
- $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

  So: $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 0$.

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Example 2. Let P be the projection matrix corresponding to orthogonal projection onto the subspace V. What are the eigenvalues and eigenvectors of P?

Solution.

- For every vector $\boldsymbol{x}$ in V, $P\boldsymbol{x} = \boldsymbol{x}$.

  These are the eigenvectors with eigenvalue 1.

- For every vector $\boldsymbol{x}$ orthogonal to V, $P\boldsymbol{x} = \boldsymbol{0}$.

  These are the eigenvectors with eigenvalue 0.

Definition 3. Given λ, the set of all eigenvectors with eigenvalue λ is called the **eigenspace** of A corresponding to λ.

Example 4. (continued) We saw that the projection matrix P has the two eigenvalues $\lambda = 0, 1$.

- The eigenspace of $\lambda = 1$ is V.
- The eigenspace of $\lambda = 0$ is $V^{\perp}$.

How to solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \iff\ & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \iff\ & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This has a nonzero solution $\iff \det(A - \lambda I) = 0$

Recipe. To find eigenvectors and eigenvalues of A.

- First, find the eigenvalues λ using:

  λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$

- Then, for each eigenvalue λ, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

Example 5. Find the eigenvectors and eigenvalues of

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Solution.

- $A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$
- $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$

  $= \lambda^2 - 6\lambda + 8 = 0 \implies \lambda_1 = 2,\ \lambda_2 = 4$

  This is the **characteristic polynomial** of A. Its roots are the eigenvalues of A.
- Find eigenvectors with eigenvalue $\lambda_1 = 2$:

  $A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\left(A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\right)$

  Solutions to $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$ have basis $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  So: $\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 2$.

  All other eigenvectors with $\lambda = 2$ are multiples of $\boldsymbol{x}_1$.

  $\operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is the **eigenspace** for eigenvalue $\lambda = 2$.
- Find eigenvectors with eigenvalue $\lambda_2 = 4$:

  $A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $\left(A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\right)$

  Solutions to $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$ have basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  So: $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 4$.

  The eigenspace for eigenvalue $\lambda = 4$ is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Example 6. Find the eigenvectors and the eigenvalues of

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution.

- The characteristic polynomial is:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda)(2-\lambda)$$

- A has eigenvalues $2, 3, 6$. $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$

The eigenvalues of a triangular matrix are its diagonal entries.

- $\lambda_1 = 2$:

  $(A - \lambda_1 I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}$

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- $\lambda_2 = 3$:

  $$(A - \lambda_2 I)\boldsymbol{x} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -1 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- $\lambda_3 = 6$:

  $$(A - \lambda_3 I)\boldsymbol{x} = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & -4 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

- In summary, A has eigenvalues $2, 3, 6$ with corresponding eigenvectors $\begin{bmatrix} 2 \\ -5/2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}$.

  These three vectors are independent. By the next result, this is always so.

Theorem 7. If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are eigenvectors of A corresponding to different eigenvalues, then they are independent.

Why?

Suppose, for contradiction, that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are dependent.

By kicking out some of the vectors, we may assume that there is (up to multiples) only one linear relation: $c_1\boldsymbol{x}_1 + \ldots + c_m\boldsymbol{x}_m = \boldsymbol{0}$.

Multiply this relation with A:

$A(c_1\boldsymbol{x}_1 + \ldots + c_m\boldsymbol{x}_m) = c_1\lambda_1\boldsymbol{x}_1 + \ldots + c_m\lambda_m\boldsymbol{x}_m = \boldsymbol{0}$

This is a second independent relation! Contradiction.

Practice problems

Example 8. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$.

Example 9. What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$?

No calculations!

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