Review

- If $Ax = \lambda x$, then x is an **eigenvector** of A with **eigenvalue** λ.

  All eigenvectors (plus $\mathbf{0}$) with eigenvalue λ form the **eigenspace** of λ.

- λ is an eigenvalue of $A \iff \underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0.$

  Why? Because $Ax = \lambda x \iff (A - \lambda I)x = \mathbf{0}$.

  By the way: this means that the eigenspace of λ is just $\mathrm{Nul}(A - \lambda I)$.

- E.g., if $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$ then $\det(A - \lambda I) = (3 - \lambda)(6 - \lambda)(2 - \lambda)$.

- Eigenvectors $x_1, \ldots, x_m$ of A corresponding to different eigenvalues are independent.

- By the way:
  - product of eigenvalues = determinant
  - sum of eigenvalues = "trace" (sum of diagonal entries)

Example 1. Find the eigenvalues of A as well as a basis for the corresponding eigenspaces, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

Solution.

- The characteristic polynomial is:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)[(3-\lambda)^2 - 1] \\ &= (2-\lambda)(\lambda-2)(\lambda-4) \end{aligned}$$

- A has eigenvalues $2, 2, 4$. $\quad A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

  Since $\lambda = 2$ is a double root, it has **(algebraic) multiplicity** 2.

- $\lambda_1 = 2$:

  $$(A - \lambda_1 I)x = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} x = \mathbf{0}$$

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Two independent solutions: $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

In other words: the eigenspace for $\lambda = 2$ is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

- $\lambda_2 = 4$:

  $(A - \lambda_2 I)\boldsymbol{x} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

- In summary, A has eigenvalues 2 and 4:
  - eigenspace for $\lambda = 2$ has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$,
  - eigenspace for $\lambda = 4$ has basis $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

An $n \times n$ matrix A has up to n different eigenvalues.

Namely, the roots of the degree n characteristic polynomial $\det(A - \lambda I)$.

- For each eigenvalue λ, A has at least one eigenvector.

  That's because $\operatorname{Nul}(A - \lambda I)$ has dimension at least 1.

- If λ has multiplicity m, then A has up to m (independent) eigenvectors for λ.

Ideally, we would like to find a total of n (independent) eigenvectors of A.

Why can there be no more than n eigenvectors?!

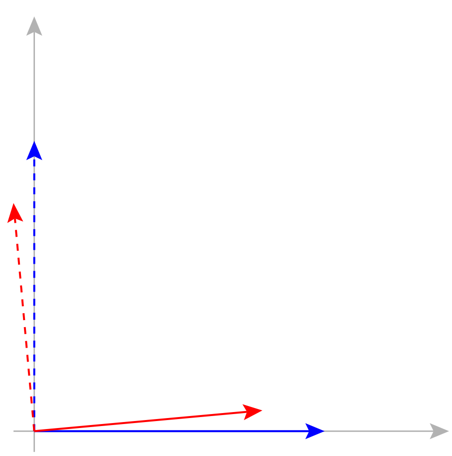
Two sources of trouble: eigenvalues can be

- complex numbers (that is, not enough real roots), or
- repeated roots of the characteristic polynomial.

Example 2. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Geometrically, what is the trouble?

Solution. $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

i.e. multiplication with A is rotation by $90°$ (counter-clockwise).



Which vector is parallel after rotation by $90°$? Trouble.

Fix: work with complex numbers!

- $\det(A-\lambda I)=\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix}=\lambda^2+1$

  So, the eigenvalues are $\lambda_1=i$ and $\lambda_2=-i$.

- $\lambda_1=i$: $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\boldsymbol{x}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{x}_1=\begin{bmatrix} i \\ 1 \end{bmatrix}$

  Let us check: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ i \end{bmatrix}=i\begin{bmatrix} i \\ 1 \end{bmatrix}$

- $\lambda_2=-i$: $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\boldsymbol{x}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \boldsymbol{x}_2=\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Example 3. Find the eigenvectors and eigenvalues of $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What is the trouble?

Solution.

- $\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix}=(1-\lambda)^2$

  So: $\lambda=1$ is the only eigenvalue (it has multiplicity 2).

- $(A-\lambda I)\boldsymbol{x}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}=\boldsymbol{0} \implies \boldsymbol{x}_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

  So: the eigenspace is $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. Only dimension 1!

- Trouble: only 1 independent eigenvector for a 2×2 matrix

  This kind of trouble cannot really be fixed.

  We have to lower our expectations and look for *generalized eigenvectors*.

  These are solutions to $(A-\lambda I)^2\boldsymbol{x}=\boldsymbol{0}$, $(A-\lambda I)^3\boldsymbol{x}=\boldsymbol{0}$, ...

Practice problems

Example 4. Find the eigenvectors and eigenvalues of $A=\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.