Review

Let A be $n \times n$ with independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

Then A can be **diagonalized** as $A = PDP^{-1}$.

- the columns of P are the eigenvectors
- the diagonal matrix D has the eigenvalues on the diagonal

Why? We need to see that $AP = PD$:

$$A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i \quad \Longrightarrow \quad A\begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

- The **differential equation** $y' = ay$ with **initial condition** $y(0) = C$ is solved by $y(t) = Ce^{at}$.

  Recall from Calculus the Taylor series $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

- Goal: similar treatment of systems like:

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Definition 1. Let A be $n \times n$. The **matrix exponential** is

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

Then: $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$

Why?
$$\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = \frac{\mathrm{d}}{\mathrm{d}t}\left(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) = A + \frac{1}{1!}A^2t + \frac{1}{2!}A^3t^2 + \cdots = Ae^{At}$$

The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

Why? Because $\boldsymbol{y}'(t) = Ae^{At}\boldsymbol{y}_0 = A\boldsymbol{y}(t)$ and $\boldsymbol{y}(0) = e^{0A}\boldsymbol{y}_0 = \boldsymbol{y}_0$.

Example 2. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$, then:

$$e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 2^2 & 0 \\ 0 & 5^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 & 0 \\ 0 & e^5 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 5t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (5t)^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

Clearly, this works to obtain e^D for any diagonal matrix D.

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Example 3. Suppose $A = PDP^{-1}$. Then, what is A^n?

Solution.

First, note that $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$.

Likewise, $A^n = PD^nP^{-1}$.

(The point being that D^n is trivial to compute because D is diagonal.)

Theorem 4. Suppose $A = PDP^{-1}$. Then, $e^A = Pe^DP^{-1}$.

Why? Recall that $A^n = PD^nP^{-1}$.

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= I + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots \\ &= P\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right)P^{-1} = Pe^DP^{-1} \end{aligned}$$

Example 5. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution. The solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0$.

- Diagonalize $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$:
  - $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$, so the eigenvalues are ± 1
  - $\lambda = 1$ has eigenspace $\operatorname{Nul}\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$
  - $\lambda = -1$ has eigenspace $\operatorname{Nul}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$
  - Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t \\ -e^{-t} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix} \end{aligned}$$

Example 6. Solve the differential equation

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution.

- Recall that the solution to $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ is $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$.
- A has eigenvalues 2 and 4. (We did that in an earlier class!)
  - $\lambda = 2$: $\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \Longrightarrow$ eigenspace $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$
  - $\lambda = 4$: $\begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \Longrightarrow$ eigenspace $\operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$
- $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 4 \end{bmatrix}$
- Compute the solution $\boldsymbol{y} = e^{At}\boldsymbol{y}_0$:

$$\begin{aligned} \boldsymbol{y} = e^{At}\boldsymbol{y}_0 &= Pe^{Dt}P^{-1}\boldsymbol{y}_0 \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 0 \\ e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix} \end{aligned}$$

Check (optional) that $\boldsymbol{y} = \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}$ indeed solves the original problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} + 4e^{4t} \\ 4e^{4t} \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} + e^{4t} \\ e^{4t} \end{bmatrix}$$

Remark 7. The matrix exponential shares many other properties of the usual exponential:

- e^A is invertible and $(e^A)^{-1} = e^{-A}$
- $e^A e^B = e^{A+B} = e^B e^A$ if $AB = BA$

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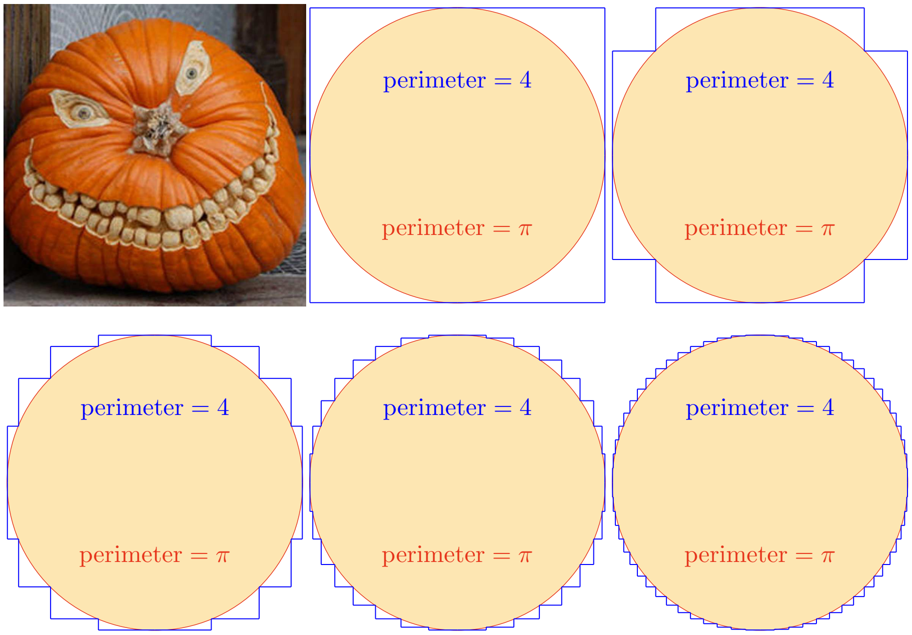
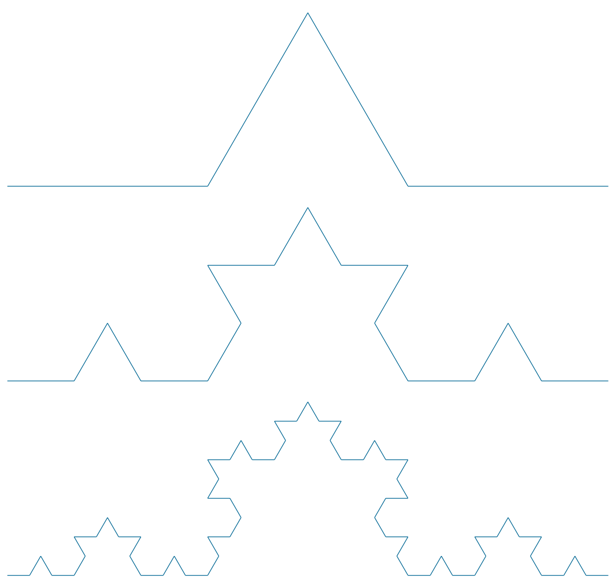
Ending the Halloween torture

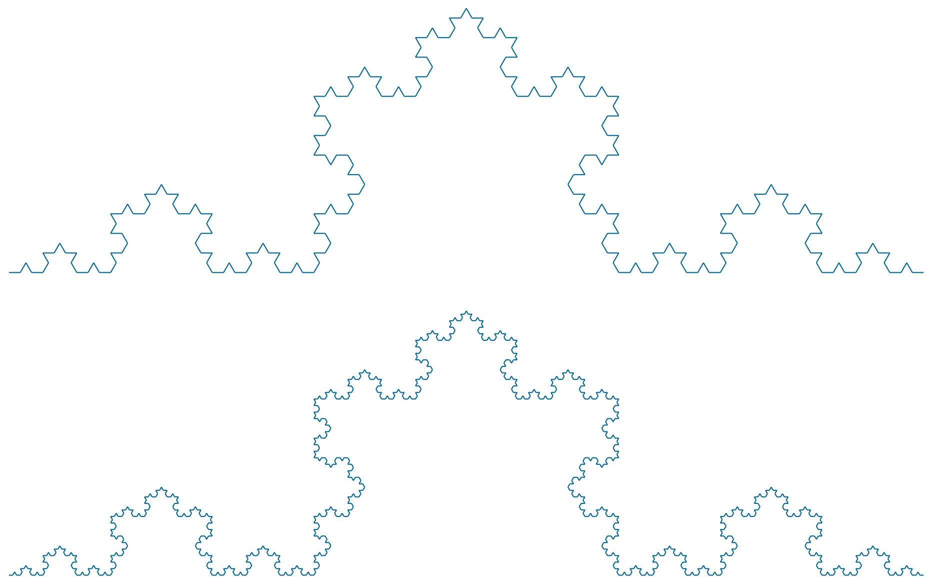


- Length of the graph of $y(x)$ on $[a, b]$ is $\int_a^b \sqrt{1 + y'(x)^2} \mathrm{d}x$.
- While the blue curve does converge to the circle, its derivative does not converge!
- In the language of functional analysis: The linear map $D: y \mapsto y'$ is not continuous!

  (That is, two functions can be close without their derivatives being close.)

Even more extreme examples are provided by **fractals**. The **Koch snowflake**:

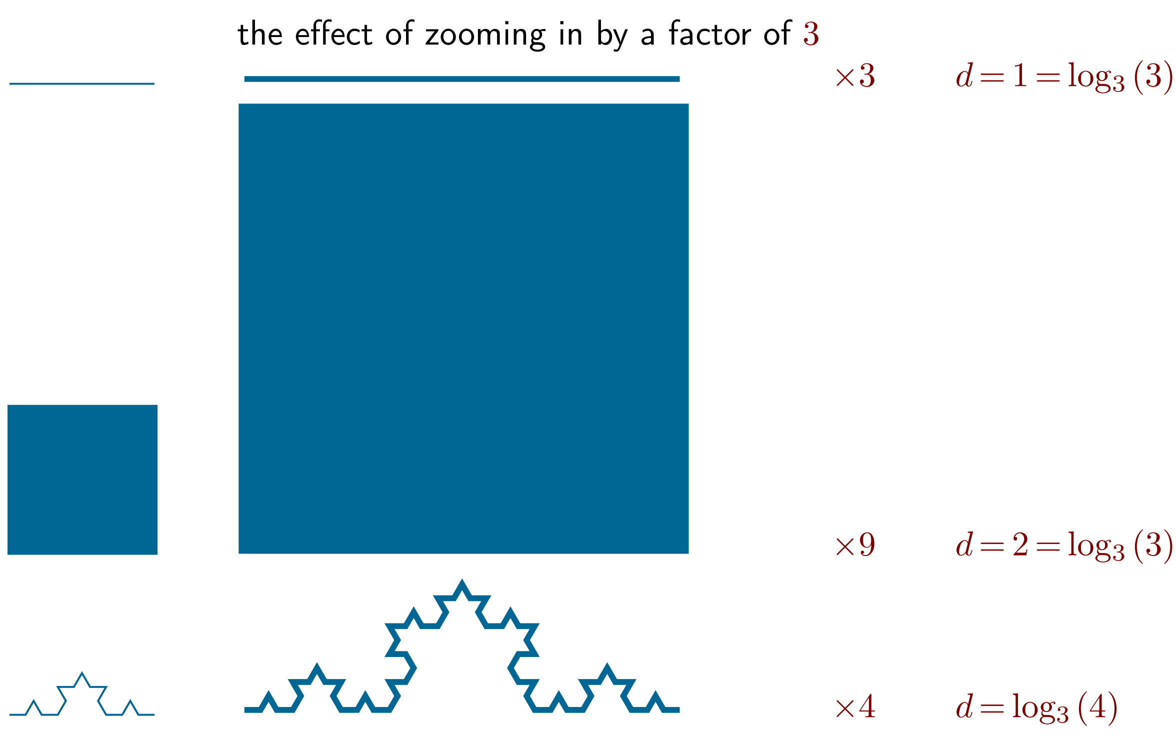




- Its perimeter is infinite!

  Why? At each iteration, the perimeter gets multiplied by $4/3$.

- Its boundary has dimension $\log_3(4) \approx 1.262$!!



- Such fractal behaviour is also observed when attempting to measure the length of a coastline: the measured length increases by a factor when using a smaller scale.

  See: `http://en.wikipedia.org/wiki/Coastline_paradox`