Preparation problems for the discussion sections on September 9th and 11th

1. *Determine if the vector* $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ *is a linear combination of* $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

Solution: We check whether there are x_1, x_2, x_3 in $\mathbb{R}$ such that

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}.$$

For this, it is enough to check whether the system of linear equations with the following augmented matrix is consistent:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right].$$

We bring the augmented matrix in echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \xrightarrow{R2 \to R2+2R1, R3 \to R3-2R1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right]$$

$$\xrightarrow{R3 \to R3-R2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The system is inconsistent, because in echelon form there is a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & x \end{array}\right],$$

where x is non-zero. Hence the vector $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is *not* a linear combination of

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}.$$

2. *Give a geometric description of* $\textbf{Span}\{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\}$.

Solution: Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ span a plane iff (this is short for "if and only if") there is no real number c such that $c\mathbf{v}_1 = \mathbf{v}_2$. Suppose there is c such that

$$c\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}.$$

By the first entry of the two vectors, we have $3c = -2$. So $c = -\frac{2}{3}$. But by the third entry, we get $2c = 3$. So $c = \frac{3}{2}$. This is impossible since $-\frac{2}{3} \neq \frac{3}{2}$. Hence $\mathbf{Span}\{\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\}$ is a plane.

3. *True or false? Justify your answers!*

(a) *Let A be an $m \times n$-matrix and B be an $m \times l$-matrix. Then the product AB is defined.*

(b) *The weights $c_1, ..., c_p$ in a linear combination $c_1\boldsymbol{v}_1 + ... + c_p\boldsymbol{v}_p$ cannot all be zero.*

(c) $\mathbf{Span}\{\boldsymbol{u}, \boldsymbol{v}\}$ *contains the line through $\boldsymbol{u}$ and the origin.*

(d) *Asking whether the linear system corresponding to $\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{b} \end{bmatrix}$ is consistent, is the same as asking whether $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.*

Solution: (a) This is false. Let A be a $m_1 \times m_2$ matrix and B be a $n_1 \times n_2$ matrix. Then AB is defined if and only if $m_2 = n_1$.
(b) This is false. The weights can be zero. Check the definition in the lecture notes!
(c) This is correct. The $\mathbf{Span}\{\boldsymbol{u}, \boldsymbol{v}\}$ contains all vectors of the form

$$c\boldsymbol{u} + 0\boldsymbol{v},$$

where c is in $\mathbb{R}$. These vector form a line through $\boldsymbol{u}$ and the origin.
(d) This is correct. Check the definition of being a linear combination in the lecture notes.

4. *Determine whether* $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ *is a linear combination of the columns of* $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$.

Solution: We check whether there are x_1, x_2, x_3 in $\mathbb{R}$ such that

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

For this, it is enough to check whether the system of linear equations with the following augmented matrix is consistent:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right].$$

We bring the augmented matrix in echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \xrightarrow{R2 \to R2 + 2R1,} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent, because in echelon form there is a row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & x \end{array}\right],$$

where x is non-zero. Hence the vector $\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the columns of $\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}$.

5. *Compute AB in two ways: (a) by the definition, where $A\boldsymbol{b}_1$ and $A\boldsymbol{b}_2$ are calculated separately, and (b) by the row-column rule for computing B.*

(i) $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ *(ii)* $A = \begin{bmatrix} 5 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$

Solution: For (i), by the row-column rule we get

$$AB = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}.$$

If we calculated $A\boldsymbol{b}_1$ and $A\boldsymbol{b}_2$ separately, we have

$$\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 13 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}.$$

For (ii), by the row-column rule we get

$$AB = \begin{bmatrix} 30 & 4 \\ 36 & 7 \end{bmatrix}.$$

If we calculated $A\boldsymbol{b}_1$ and $A\boldsymbol{b}_2$ separately, we have

$$\begin{bmatrix} 5 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 30 \\ 36 \end{bmatrix}$$

and

$$\begin{bmatrix} 5 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

6. *Let* $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

(1) *If* $\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, *what is* $A\boldsymbol{x}$*?*

(2) *If* $\boldsymbol{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, *what is* $A\boldsymbol{x}$*?*

(3) *Is* $A\boldsymbol{x} = \boldsymbol{b}$ *uniquely solvable: is there for a given* $\boldsymbol{b}$ *always exactly one* $\boldsymbol{x}$*?*

Solution: For (1):

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For (2):

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For (3): As we have shown in (1) and (2), if $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, there are two vectors $\boldsymbol{x}$, namely $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, such that $A\boldsymbol{x} = \boldsymbol{b}$ (this means that there are actually infinitely many solutions $\boldsymbol{x}$; find all of them!).

7. *(Some interesting matrices) Find a matrix A (what size!) such that*

(i) $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(ii) $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y + 3x \end{bmatrix}$

(iii) $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

Solution: For (i):

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For (ii):

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

For (iii):

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$