Preparation problems for the discussion sections on February 20th and 21st

1. *Determine which of the following sets are subspaces and* give reasons*:*

(a) $W_1 = \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \ : \ a - 2b = c, 4a + 2c = 1\}$,

(b) $W_2 = \{ \begin{bmatrix} a-b \\ c \\ a+c \\ a-2b-c \end{bmatrix} \ : \ a, b, c \in \mathbb{R}\}$,

(c) $W_3 = \{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ a \cdot b \geq 0\}$.

(d) $W_4 = \{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ a^2 + b^2 \leq 1\}$.

Solution: a) W_1 is not a subspace, since the zero vector is not in W_1. The zero vector is not in W_1, because

$$4 \cdot 0 + 2 \cdot 0 \neq 1.$$

b) Since

$$\{ \begin{bmatrix} a-b \\ c \\ a+c \\ a-2b-c \end{bmatrix} \ : \ a, b, c \in \mathbb{R}\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and every span is a subspace, this set is a subspace as well.

c) W_3 is not a vector subspace. Consider the two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Both are in W_3, because $1 \cdot 0 = 0 \cdot (-1) \geq 0$. But

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $1 \cdot (-1) = -1 < 0$. Hence $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not in W_3. Hence W_3 is not closed under addition.

d) This set is not a vector subspace. Consider the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $1^2 \leq 1$, we have that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in W_4. However, $2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is not in W_4, since $2^2 = 4 > 1$. Hence W_4 is not closed under scalar multiplication.

2. *Is* $H = \left\{ \begin{bmatrix} a+1 \\ a \end{bmatrix} : a \text{ in } \mathbb{R} \right\}$ *a subspace of* $\mathbb{R}^2$*? Why or why not?*
Is $K = \left\{ \begin{bmatrix} a+1 \\ b \end{bmatrix} : a \text{ and } b \text{ in } \mathbb{R} \right\}$ *a subspace of* $\mathbb{R}^2$*? Why or why not?*

Solution: The set H is not a subspace, because it does not contain the zero vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (Why? Because if there is a in $\mathbb{R}$ such that

$$\begin{bmatrix} a+1 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then $a+1=0$ and $a=0$. Such an a can not exist). While H is not a subspace, K is a subspace. It is enough to realize that

$$\left\{ \begin{bmatrix} a+1 \\ b \end{bmatrix} : a \text{ and } b \text{ in } \mathbb{R} \right\} = \left\{ \begin{bmatrix} c \\ b \end{bmatrix} : c \text{ and } b \text{ in } \mathbb{R} \right\} = \text{span}\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} = \mathbb{R}^2.$$

Note that the first equality holds, because you can take c to be $a-1$.

3. *Is the set* H *of all matrices of the form* $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ *a subspace of* $M_{2\times 2}$*? Explain.*

Solution: Let A be a matrix in H. There are a, b in $\mathbb{R}$ such that

$$A = \begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = a \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

Hence A is a linear combination (in $M_{2\times 2}$) of $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$. Hence

$$H = \text{span}\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \},$$

and so H is a subspace of $M_{2\times 2}$.

4. *A matrix* B *is called symmetric if* $B^T = B$*. Let* V *be the set of all symmetric* 2×2*-matrices. Is* V *a subspace of* $M_{2\times 2}$*?*

Solution: Yes, V is a subspace of $M_{2\times 2}$. We have to check that it contains $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and is closed under addition and scalar multiplication. First note that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in V, because it is obviously symmetric.

Now take two matrices A, B in V. So we have $A^T = A$ and $B^T = B$. Then we have

$$(A+B)^T = A^T + B^T = A + B.$$

Hence $A+B$ is in V. So V is closed under addition.

Now take a matrix A in V and a scalar r. Since A is in V, we have $A^T = A$. Then we have

$$(rA)^T = rA^T = rA.$$

Hence rA is in V. So V is closed under scalar multiplication.