Preparation problems for the discussion sections on September 30th and October 2nd

1. *Find an explicit description of Nul A, where*

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

Solution: We first bring the augmented matrix of the equation $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$ into reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right] \xrightarrow{R1 \to R1 - 3R2} \left[\begin{array}{cccc|c} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right]. \tag{0.1}$$

Since the first two columns of the augmented matrix are pivot columns, the variables x_1, x_2 are basic and the variables x_3, x_4 are free. From (0.1), we get

$$x_1 = 7x_3 - 6x_4, \text{ and } x_2 = -4x_3 + 2x_4.$$

Hence any solution of $Ax = 0$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\text{Nul}(A) = \text{Span}(\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \}).$$

2. *Let* $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ *and* $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.
Find the set of all solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$, *and express it as the sum of a particular solution and solutions in the null space of A.*

Solution: **Step 1:** We start by bringing the augmented matrix of $Ax = b$ into echelon form.

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - 2R1} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since columns $1, 3$ are the pivot columns, x_2, x_4 are the free variables.

Step 2: Find a particular solution to $Ax = b$.
In order to find a particular solution to $Ax = b$, set $x_2 = x_4 = 0$. Then by the second row of the echelon form, $2x_3 = 1$ and hence $x_3 = 0.5$. Hence the first row gives

$$1x_1 + 0.5 = 1.$$

Hence $\begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$ is a particular solution of $Ax = b$.

Step 3: Find an explicit description of $\text{Nul}(A)$.

The echelon form of the augmented matrix of $Ax = 0$ is

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We will now bring it to reduced echelon form.

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R2 \to R2/2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\xrightarrow{R1 \to R1 - R2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

we get

$$x_1 = -3x_2, \text{ and } x_3 = -2x_4.$$

Hence any solution of $Ax = 0$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Hence

$$\text{Nul}(A) = \text{Span}(\{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\}).$$

Step 4: Adding the null space to the particular solution.

Every solution of $Ax = b$ is of the form

$$\begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix},$$

where c_1, c_2 are real numbers.

3. *Consider the subspace*

$$W := \{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;:\; a = 2b + c,\; 2a = c - 3d\}.$$

Find a matrix A and a matrix B such that $W = Col(A)$ and $W = Nul(B)$.

Solution: Let B be

$$\begin{bmatrix} -1 & 2 & 1 & 0 \\ -2 & 0 & 1 & -3 \end{bmatrix}.$$

It is easy to see that $B\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ iff $a = 2b + c$ and $2a = c - 3d$. Hence $W = Nul(B)$.

Note that $a = 2b + c$ and $2a = c - 3d$ hold iff $a = 2b + c$ and $d = -\frac{4}{3}b - \frac{1}{3}c$. Hence for $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in W, we have

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2b + c \\ b \\ c \\ -\frac{4}{3}b - \frac{1}{3}c \end{bmatrix} = b\begin{bmatrix} 2 \\ 1 \\ 0 \\ -\frac{4}{3} \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \\ -\frac{1}{3} \end{bmatrix}.$$

Hence

$$W = Col(\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \\ -\frac{4}{3} & -\frac{1}{3} \end{bmatrix}).$$

[Note that your answer might look different. In that case, as an exercise, check that the spaces are indeed the same. By the way, if you like recipes, you can always proceed as in the very first problem to express any null space as a column space.]

4. *a) For which values of h is $\boldsymbol{v}_3$ in the span of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$? b) For which values of h is $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ linearly dependent?*

(i) $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 10 \\ 6 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ -9 \\ h \end{bmatrix},$

(ii) $\boldsymbol{v}_1 = \begin{bmatrix} -7 \\ 3 \\ -6 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ h \end{bmatrix},$

(iii) $\boldsymbol{v}_1 = \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 6 \\ -12 \\ 2 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}.$

Solution: We bring the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ to echelon form.

For (i),

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \xrightarrow{R2\to R2+5R1, R3\to R3+3R1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}$$
$$\xrightarrow{R3\to R3-(h+6)R2} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This can never have three pivots. Hence for every h, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Because of R2, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ is inconsistent, independent of what h is. In other words, $\mathbf{v}_3$ is never in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

For (ii),

$$\begin{bmatrix} -7 & 2 & 1 \\ 3 & -1 & 2 \\ -6 & 4 & h \end{bmatrix} \xrightarrow{R2\to R2+3/7R1, R3\to R3-6/7R1} \begin{bmatrix} -7 & 2 & 1 \\ 0 & -1/7 & 17/7 \\ 0 & 16/7 & h-6/7 \end{bmatrix}$$
$$\xrightarrow{R3\to R3+16R2} \begin{bmatrix} -7 & 2 & 1 \\ 0 & -1/7 & 17/7 \\ 0 & 0 & h+38 \end{bmatrix}$$

For (a), this system is inconsistent iff $h \neq -38$. Hence $\mathbf{v}_3$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ iff $h = -38$.
For (b), the matrix has three pivots iff $h \neq -38$. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent iff $h \neq -38$.

For (iii),

$$\begin{bmatrix} 6 & 6 & 9 \\ -4 & -12 & h \\ 3 & 2 & 3 \end{bmatrix} \xrightarrow{R2\to R2+2/3R1, R3\to R3-1/2R1} \begin{bmatrix} 6 & 6 & 9 \\ 0 & -8 & h+6 \\ 0 & -1 & -3/2 \end{bmatrix}$$
$$\xrightarrow{R3\to R3-1/8R2} \begin{bmatrix} 6 & 6 & 9 \\ 0 & -8 & h+6 \\ 0 & 0 & \frac{-h-18}{8} \end{bmatrix}$$

For (a), this system is inconsistent iff $h \neq -18$. Hence $\mathbf{v}_3$ in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ iff $h = -18$.
For (b), the matrix has three pivots iff $h \neq -18$. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent iff $h \neq -18$.

5. *Check whether the following sets of vectors are linearly independent. Justify your answer!*

a) $\left\{ \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} \right\}$ *b)* $\left\{ \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ *d)* $\left\{ \begin{bmatrix} 4 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ -11 \end{bmatrix} \right\}$

Solution: (a) Linearly dependent, since $\frac{3}{2}\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$.
(b) Linearly dependent, because there are four vectors, but only two entries in each vector.
(c) Linearly dependent, because it contains the zero vector.
(d) Linearly independent. Indeed:

$$\begin{bmatrix} 4 & 0 & 3 \\ 0 & -5 & 1 \\ -1 & 5 & -1 \\ 2 & 10 & -11 \end{bmatrix} \xrightarrow{R3 \to R3 + 1/4R1, R4 \to R4 - 1/2R1} \begin{bmatrix} 4 & 0 & 3 \\ 0 & -5 & 1 \\ 0 & 5 & -1/4 \\ 0 & 10 & -25/2 \end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + R2, R4 \to R4 + 2R2} \begin{bmatrix} 4 & 0 & 3 \\ 0 & -5 & 1 \\ 0 & 0 & 3/4 \\ 0 & 0 & -21/2 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 + 42/3R3} \begin{bmatrix} 4 & 0 & 3 \\ 0 & -5 & 1 \\ 0 & 0 & 3/4 \\ 0 & 0 & 0 \end{bmatrix}$$

There is a pivot position in every column. Hence the vectors are linearly independent.

6. *True or false? Justify your answers!*

(a) *If three vectors in $\mathbb{R}^3$ span a plane, then they are linearly dependent.*
(b) *If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.*
(c) *If $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent, and if $\boldsymbol{z}$ is in the span of $\boldsymbol{x}$ and $\boldsymbol{y}$, then $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly dependent.*
(d) *If a set in $\mathbb{R}^n$ is linearly independent, then it contains n vectors.*

Solution:(a) True. That just means that one of them has to be a linear combination of the other two. Hence they have to be linearly dependent.
(b) False. $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\}$ is linearly dependent.
(c) True. If $c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{z}$, then $c_1\mathbf{x} + c_2\mathbf{y} - \mathbf{z} = 0$. So there is a non-trivial linear combination.
(d) False. $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$ is linearly independent.