Preparation problems for the discussion sections on October 28th and 30th

1. *Let* $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. *Let* $\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. *Can you find real numbers* c_1, c_2 *such that*

$$\boldsymbol{v} = c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2?$$

2. *Let* $W = \text{Span}\{\boldsymbol{v}\}$, *where* $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, *be a subspace of* $\mathbb{R}^3$. *Find the projections* $\boldsymbol{a}_W, \boldsymbol{b}_W, \boldsymbol{c}_W$ *of the vectors*

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

onto the subspace W. *Interpret your results geometrically.*

3. *Let* $W = \text{Span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\}$ *be a subspace of* $\mathbb{R}^4$.

(i) *Find the closest point to* $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *on the subspace* W.

(ii) *Find the projection matrix,* P, *corresponding to the projection onto* W.

(iii) *Use the projection matrix,* P, *to find the projection of* $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ *onto the subspace* W.

4. *Let* $W = \text{Span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\}$ *and* $V = \text{Span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\}$ *be subspaces of* $\mathbb{R}^3$.

(i) *Find the projection matrices,* P *and* Q, *corresponding to the projections onto* W *and* V, *respectively.*

(ii) *Check that* $PQ = QP$. *Can you explain this by interpreting* PQ *as a projection matrix?*

5. *Let* $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $\boldsymbol{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

a. *Does* $\boldsymbol{b}$ *belong to the column space of* A? *Can you solve* $A\boldsymbol{x} = \boldsymbol{b}$?

b. *What do you expect the projection of* $\boldsymbol{b}$ *onto* $W = Col(A)$ *to be?*

c. *Find the projection* $\widehat{b}$ *of* $\boldsymbol{b}$ *onto* $Col(A)$, *and then solve* $A\widehat{x} = \widehat{b}$. *(The vector* $\widehat{x}$ *is called* the least square solution *of* $A\boldsymbol{x} = \boldsymbol{b}$.*)*

d. *Solve the equation* $A^T A\widehat{x} = A^T\boldsymbol{b}$. *Compare with your result of the previous part! (This equation is called the* normal equation *of* $A\boldsymbol{x} = \boldsymbol{b}$.*)*

e. *Answer these questions for* A *as above but with* $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *(and then* $\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$*).*