Math 415 - Midterm 3

Thursday, November 20, 2014

Circle your section:

Philipp Hieronymi	2pm	3pm
Armin Straub	9am	11am

Name:

NetID:

UIN:

Problem 0. *[1 point]* Write down the number of your discussion section (for instance, AD2 or ADH) and the first name of your TA (Allen, Anton, Mahmood, Michael, Nathan, Pouyan, Tigran, Travis).

Section:	TA:

To be completed by the grader:

0	1	2	3	4	5	6	Shorts	$\sum$
/1	/?	/?	/?	/?	/?	/?	/?	/?

Good luck!

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of ? pages. Take a moment to make sure you have all pages.
- You have 75 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page (make it clear if you do).
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to **write down all row operations** for full credit.

Problem 1. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix}$. Find a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Solution. We have to solve $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

and,

$$A^T\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

Since

$$\left[\begin{array}{cc|c} 4 & 2 & 20 \\ 2 & 6 & 20 \end{array}\right] \xrightarrow{R2 \to R2 - 1/2R1} \left[\begin{array}{cc|c} 4 & 2 & 20 \\ 0 & 5 & 10 \end{array}\right],$$

we obtain

$$\hat{\boldsymbol{x}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Problem 2. Let $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find an orthonormal basis for W.

(b) What is the orthogonal projection of $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ onto W?

(c) Write $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ as the sum of a vector in W and a vector in $W^\perp$.

(d) Find the projection matrix corresponding to orthogonal projection onto W.

Solution.

(a) We apply Gram-Schmidt to $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. We have:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}{\| \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

and,

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1}{\|\mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1\|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - (\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}}{\| \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - (\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}) \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \|} = \frac{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}{\| \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \|} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence, $\left\{ \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis for W.

(b) Using the orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for W, we find that the orthogonal projection of $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ onto W is

$$(\mathbf{w} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{w} \cdot \mathbf{u}_2)\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is orthogonal to W, therefore we have:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is orthogonal to W and $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ is in W. (Hence, the projection of $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ onto W is $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$.)

(d) We know that $W = \text{Col}(\begin{bmatrix}0 & 0\\ \frac{1}{\sqrt{2}} & 0\\ 0 & 1\\ \frac{1}{\sqrt{2}} & 0\end{bmatrix})$. Since the columns of $Q = \begin{bmatrix}0 & 0\\ \frac{1}{\sqrt{2}} & 0\\ 0 & 1\\ \frac{1}{\sqrt{2}} & 0\end{bmatrix}$ are orthonormal, the projection matrix onto W is:

$$QQ^T = \begin{bmatrix}0 & 0\\ \frac{1}{\sqrt{2}} & 0\\ 0 & 1\\ \frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$$

Here, we used Problem 4(a) from discussion problem set 11. However, you do not need to know this formula. An alternative way to compute the projection matrix is to project the standard unit vectors onto W. The results are the columns of the projection matrix. Observe how, in particular, the first column matches the observation in the previous problem that the first standard basis vector gets projected to zero.

[Note: once we have the projection matrix, we can also find the projection in part (b) by multiplication with the projection matrix, i.e., $\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 0 & 1 & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\2\\1\\0\end{bmatrix}$.]

Problem 3. Find the QR decomposition of $A = \begin{bmatrix}4 & 25 & 0\\ 0 & 0 & -2\\ 3 & -25 & 0\end{bmatrix}$.

Solution. We start with the columns of $A(= [\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3])$ and we use Gram-Schmidt to find the columns of $Q(= [\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3])$:

$$q_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{\begin{bmatrix}4\\0\\3\end{bmatrix}}{\|\begin{bmatrix}4\\0\\3\end{bmatrix}\|} = \begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix}$$

and,

$$\mathbf{q}_2 = \frac{\mathbf{v}_2 - (\mathbf{q}_1\cdot\mathbf{v}_2)\mathbf{q}_1}{\|\mathbf{v}_2 - (\mathbf{q}_1\cdot\mathbf{v}_2)\mathbf{q}_1\|} = \frac{\begin{bmatrix}25\\0\\-25\end{bmatrix} - (\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix}\cdot\begin{bmatrix}25\\0\\-25\end{bmatrix})\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix}}{\|\begin{bmatrix}25\\0\\-25\end{bmatrix} - (\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix}\cdot\begin{bmatrix}25\\0\\-25\end{bmatrix})\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix}\|} = \frac{\begin{bmatrix}21\\0\\-28\end{bmatrix}}{\|\begin{bmatrix}21\\0\\-28\end{bmatrix}\|} = \begin{bmatrix}\frac{21}{35}\\0\\-\frac{28}{35}\end{bmatrix} = \begin{bmatrix}\frac{3}{5}\\0\\-\frac{4}{5}\end{bmatrix}$$

and,

$$\mathbf{q}_3 = \frac{\mathbf{v}_3 - (\mathbf{q}_1\cdot\mathbf{v}_3)\mathbf{q}_1 - (\mathbf{q}_2\cdot\mathbf{v}_3)\mathbf{q}_2}{\|\mathbf{v}_3 - (\mathbf{q}_1\cdot\mathbf{v}_3)\mathbf{q}_1 - (\mathbf{q}_2\cdot\mathbf{v}_3)\mathbf{q}_2\|} = \frac{\begin{bmatrix}0\\-2\\0\end{bmatrix} - (\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix}\cdot\begin{bmatrix}0\\-2\\0\end{bmatrix})\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix} - (\begin{bmatrix}\frac{3}{5}\\0\\-\frac{4}{5}\end{bmatrix}\cdot\begin{bmatrix}0\\-2\\0\end{bmatrix})\begin{bmatrix}\frac{3}{5}\\0\\-\frac{4}{5}\end{bmatrix}}{\|\begin{bmatrix}0\\-2\\0\end{bmatrix} - (\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix}\cdot\begin{bmatrix}0\\-2\\0\end{bmatrix})\begin{bmatrix}\frac{4}{5}\\0\\\frac{3}{5}\end{bmatrix} - (\begin{bmatrix}\frac{3}{5}\\0\\-\frac{4}{5}\end{bmatrix}\cdot\begin{bmatrix}0\\-2\\0\end{bmatrix})\begin{bmatrix}\frac{3}{5}\\0\\-\frac{4}{5}\end{bmatrix}\|}$$

$$= \frac{\begin{bmatrix}0\\-2\\0\end{bmatrix}}{\|\begin{bmatrix}0\\-2\\0\end{bmatrix}\|} = \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

Hence,

$$Q = \begin{bmatrix}\frac{4}{5} & \frac{3}{5} & 0\\ 0 & 0 & -1\\ \frac{3}{5} & -\frac{4}{5} & 0\end{bmatrix}$$

Finally:

$$R = Q^T A = \begin{bmatrix}\frac{4}{5} & 0 & \frac{3}{5}\\ \frac{3}{5} & 0 & -\frac{4}{5}\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}4 & 25 & 0\\ 0 & 0 & -2\\ 3 & -25 & 0\end{bmatrix} = \begin{bmatrix}5 & 5 & 0\\ 0 & 35 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Problem 4. Find the least squares line for the data points $(1,1)$, $(2,1)$, $(3,4)$, $(4,4)$.

Solution. We have to find a and b (where the least squares line is $y = ax + b$) so that $\begin{bmatrix}a\\b\end{bmatrix}$ is the least squares solution of:

$$A\boldsymbol{x} = \begin{bmatrix}1 & 1\\ 2 & 1\\ 3 & 1\\ 4 & 1\end{bmatrix}\boldsymbol{x} = \begin{bmatrix}1\\1\\4\\4\end{bmatrix} = \boldsymbol{b}$$

We have to solve $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$:

$$A^T A = \begin{bmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 1\\ 3 & 1\\ 4 & 1\end{bmatrix} = \begin{bmatrix}30 & 10\\ 10 & 4\end{bmatrix} \qquad A^T\boldsymbol{b} = \begin{bmatrix}1 & 2 & 3 & 4\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\4\\4\end{bmatrix} = \begin{bmatrix}31\\10\end{bmatrix}$$

we have:

$$\left[\begin{array}{cc|c}30 & 10 & 31\\ 10 & 4 & 10\end{array}\right] \xrightarrow{R1\to R1-3R2,\, R1\leftrightarrow R2} \left[\begin{array}{cc|c}10 & 4 & 10\\ 0 & -2 & 1\end{array}\right]$$

Hence,

$$\hat{\boldsymbol{x}} = \begin{bmatrix}\frac{6}{5}\\ -\frac{1}{2}\end{bmatrix}$$

Therefore, the least squares line for the given data is $y = \frac{6}{5}x - \frac{1}{2}$.

Problem 5.

(a) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Write down the cofactor expansion of $\det(A)$ along the second column.

(b) Let $A = [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3]$ and $B = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]$ be two 3×3-matrices. Suppose that $\det(A) = 5$ and $\boldsymbol{b}_1 = \boldsymbol{a}_1$, $\boldsymbol{b}_2 = \boldsymbol{a}_1 + 2\boldsymbol{a}_2$, $\boldsymbol{b}_3 = \boldsymbol{a}_3$. What is $\det(B)$?

(c) Find $\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}$.

(d) Find $\det \begin{bmatrix} 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$.

Solution.

(a) We have:

$$\det(A) = -b\det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + e\det(\begin{bmatrix} a & c \\ g & i \end{bmatrix}) - h\det(\begin{bmatrix} a & c \\ d & f \end{bmatrix})$$

(b) We have:

$$A \xrightarrow{C2 \to 2C2, C2 \to C2 + C1} B$$

The first column operation multiplies the determinant by 2, and the second column operation does not change the value of the determinant. Hence,

$$\det(B) = 2\det(A) = 10.$$

Note: Since $\det(A) = \det(A^T)$, we have the same rules for column operations as we are used to for row operations. (That's because a column operation on A is equivalent to a row operation on A^T.)

(c) We use row operations to transform the matrix into an upper triangular matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix} \xrightarrow{R2 \to R2 - R1, R3 \to R3 - R1, R4 \to R4 - R1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{bmatrix}$$

$$\xrightarrow{R4 \to R4 - R3, R3 \leftrightarrow R2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{bmatrix} \xrightarrow{R4 \to R4 - 2R3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

Since we swap rows once, and the other row operations that we used do not change the value of the determinant, we have:

$$\det(A) = -[1.(-2).3.(-12)] = -72$$

(d) We use row operations to transform the matrix into an upper triangular matrix:

$$A = \begin{bmatrix} 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R4, R1 \to R1 - 3/2R2} \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & -1 & 3 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R4, R2 \leftrightarrow R3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Since we swap rows twice, and the other row operations that we used do not change the value of the determinant, we have:

$$\det(A) = -(-[1.(-1).2.(-2)]) = 4$$

[Note that this determinant is also pleasant to compute by expanding along the second column. Do it!]

Problem 6. Let $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) Find the eigenvalues of A, as well as a basis for the corresponding eigenspaces.
(b) Diagonalize A. (That is, write $A = PDP^{-1}$ where D is diagonal.)

Solution.

(a) We have:

$$\det \begin{bmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda)(-1-\lambda)$$

Hence, the eigenvalues of A are 2,4, and -1. For $\lambda = 2$:

$$\begin{bmatrix} -3 & 0 & 1 \\ -3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \to R2-R1, R2 \to 1/2R2, R1 \to -1/3R1} \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$.

For $\lambda = 4$:

$$\begin{bmatrix} -5 & 0 & 1 \\ -3 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R3 \to -1/2R3, R2 \to R2-R3, R1 \to R1-R3, R2 \to R2-3/5R1, R3 \to -1/5R3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = -1$:

$$\begin{bmatrix} 0 & 0 & 1 \\ -3 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{R2 \to R2-R1, R3 \to R3-3R1, R2 \to -1/3R2} \begin{bmatrix} 0 & 0 & 1 \\ 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the corresponding eigenspace is span $\left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} \right\}$.

(b) The columns of P are (linearly independent) eigenvectors of A and D is the diagonal matrix with eigenvalues of A on the main diagonal in the appropriate order (corresponding to columns of P). Therefore:

$$P = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 3 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Problem 7. Consider the vector space

$$V = \{f : \mathbb{R} \to \mathbb{R} \ : \ f \text{ is 7-periodic and } f \text{ is "nice"}\}.$$

Here, f "nice" means, for instance, that f should be piecewise continuous or (more generally) that $\int_0^7 f(t)^2 dt$ should be finite.

(a) What is a natural inner product on V?

(b) Consider the 7-periodic function $f(t)$ with $f(t) = \begin{cases} 1, & \text{for } 0 \le t < 3, \\ 2, & \text{for } 3 \le t < 7. \end{cases}$

Compute the orthogonal projection of $f(t)$ onto the span of $\cos\left(3\frac{2\pi t}{7}\right)$.

Solution.

(a) A natural inner product on V is:

$$(f, g) = \int_0^7 f(t)g(t)dt$$

(b) The orthogonal projection of $f(t)$ onto the span of $\cos\left(3\frac{2\pi t}{7}\right)$ is: (let $g(t) = \cos\left(3\frac{2\pi t}{7}\right)$)

$$\frac{(f,g)}{(g,g)}g = \frac{\int_0^7 f(t)g(t)dt}{\int_0^7 g(t)g(t)dt}g = \frac{\int_0^3 \cos\left(3\frac{2\pi t}{7}\right)dt + 2\int_3^7 \cos\left(3\frac{2\pi t}{7}\right)dt}{\int_0^7 \cos^2\left(3\frac{2\pi t}{7}\right)dt}g = \frac{\frac{7}{6\pi}\sin\left(\frac{6\pi t}{7}\right)|_0^3 + \frac{14}{6\pi}\sin\left(\frac{6\pi t}{7}\right)|_3^7}{\int_0^7 \frac{1+\cos\left(6\frac{2\pi t}{7}\right)}{2}dt}g$$

$$= \frac{-\frac{7}{6\pi}\sin\left(\frac{18\pi}{7}\right)}{\frac{7}{2}}g = -\frac{1}{3\pi}\sin\left(\frac{4\pi}{7}\right)\cos\left(3\frac{2\pi t}{7}\right)$$

Problem 8. Consider the space $\mathbb{P}^2$ of polynomials of degree up to 2, together with the inner product

$$\langle p(t), q(t)\rangle = \int_0^1 p(t)q(t)\mathrm{d}t.$$

(a) Is the standard basis $1, t, t^2$ an orthogonal basis?

(b) Apply Gram–Schmidt to $1, t, t^2$ to obtain an orthonormal basis of $\mathbb{P}^2$.

(c) What is the orthogonal projection of t^2 onto $\mathrm{span}\{1, t\}$?

Solution.

(a) Let us compute the inner products to find out:

$$\begin{aligned} \langle 1, t\rangle &= \int_0^1 s\mathrm{d}s = \frac{1}{2} \\ \langle 1, t^2\rangle &= \int_0^1 s^2\mathrm{d}s = \frac{1}{3} \\ \langle t, t^2\rangle &= \int_0^1 s^3\mathrm{d}s = \frac{1}{4} \end{aligned}$$

So, no, this is not an orthogonal basis.

[Note: we are using s in the integral just so we don't get confused when we write down things like $\langle t^2, t\rangle t = \int_0^1 s^3\mathrm{d}s \cdot t = \frac{1}{4}t$ during Gram–Schmidt.]

(b) In the first step, we only normalize to get

$$q_1(t) = \frac{1}{\langle 1, 1\rangle} = \frac{1}{\int_0^1 1\mathrm{d}s} = 1.$$

In the second step of Gram–Schmidt, we calculate

$$b_2(t) = t - \langle t, q_1(t)\rangle q_1(t) = t - \frac{1}{2},$$

which we normalize to get

$$q_2(t) = \frac{b_2(t)}{\sqrt{\langle b_2(t), b_2(t)\rangle}} = \frac{t-\frac{1}{2}}{\sqrt{\int_0^1 \left(s-\frac{1}{2}\right)^2 \mathrm{d}s}} = \frac{t-\frac{1}{2}}{\sqrt{\frac{1}{3}-\frac{1}{2}+\frac{1}{4}}} = \sqrt{12}\left(t-\frac{1}{2}\right).$$

In the third step of Gram–Schmidt, we calculate

$$\begin{aligned} b_3(t) &= t^2 - \langle t^2, q_1(t)\rangle q_1(t) - \langle t^2, q_2(t)\rangle q_2(t) \\ &= t^2 - \int_0^1 s^2 \mathrm{d}s \cdot 1 - \int_0^1 s^2\sqrt{12}\left(s-\frac{1}{2}\right)\mathrm{d}s \cdot \sqrt{12}\left(t-\frac{1}{2}\right) \\ &= t^2 - \frac{1}{3} - 12\int_0^1 \left(s^3 - \frac{s^2}{2}\right)\mathrm{d}s \cdot \left(t-\frac{1}{2}\right) \\ &= t^2 - \frac{1}{3} - 12\left(\frac{1}{4}-\frac{1}{6}\right)\left(t-\frac{1}{2}\right) \\ &= t^2 - t + \frac{1}{6}, \end{aligned}$$

which we normalize to get

$$\begin{aligned} q_3(t) &= \frac{b_3(t)}{\sqrt{\langle b_3(t), b_3(t)\rangle}} \\ &= \frac{t^2-t+\frac{1}{6}}{\sqrt{\int_0^1 \left(s^2-s+\frac{1}{6}\right)^2 \mathrm{d}s}} \\ &= \frac{t^2-t+\frac{1}{6}}{\sqrt{1/180}} = \sqrt{180}\left(t^2-t+\frac{1}{6}\right). \end{aligned}$$

We have found the orthonormal basis $q_1(t), q_2(t), q_3(t)$.

(c) From our previous calculation, we know that

$$\mathrm{span}\{1, t\} = \mathrm{span}\{q_1(t), q_2(t)\}.$$

Since $q_1(t)$ and $q_2(t)$ are orthonormal, the projection of t^2 onto this span is

$$\begin{aligned} &\langle t^2, q_1(t)\rangle q_1(t) + \langle t^2, q_2(t)\rangle q_2(t) \\ =& \int_0^1 s^2\mathrm{d}s \cdot 1 + \int_0^1 s^2\sqrt{12}\left(s-\frac{1}{2}\right)\mathrm{d}s \cdot \sqrt{12}\left(t-\frac{1}{2}\right) \\ =& \ t - \frac{1}{6}, \end{aligned}$$

where we did the same calculation that we did for $b_3(t)$ during Gram–Schmidt.

SHORT ANSWERS

Note: On the actual exam all short answer question will be multiple choice. You will be entering your answers to the multiple choice questions on a scantron sheet that will be included with your exam. So please bring a **Number 2 pencil** to the exam. Thanks.

Short Problem 1. If A and B are 3×3 matrices with $\det(A) = 4$ and $\det(B) = -1$. What is the determinant of $C = 2A^T A^{-1} BA$?

Solution. We have:

$$\det(C) = 2^3 \det(A^T) \det(A^{-1}) \det(B) \det(A) = 2^3 \det(A) \frac{1}{\det(A)} \det(B) \det(A)$$
$$= 8 \det(A) \det(B) = -32$$

Short Problem 2. If A is an $n \times n$ matrix, and S is an invertible $n \times n$ matrix. Are the characteristic polynomial of A and SAS^{-1} equal? The determinant?

Solution. Yes, both the characteristic polynomial and the determinant are equal:

$$\det(SAS^{-1}) = \det(S) \det(A) \frac{1}{\det(S)} = \det(A),$$

and

$$\det(SAS^{-1} - \lambda I) = \det(S(A - \lambda I)S^{-1}) = \det(A - \lambda I).$$

Short Problem 3. Let A be a 7×7 matrix with $\dim \mathrm{Nul}(A) = 1$. What can you say about $\det(A)$?

Solution. The null space of A contains a nonzero vector. Hence, $A\boldsymbol{x} = 0$ has a nonzero solution, i.e., A is not invertible. Therefore, $\det(A) = 0$.

Short Problem 4. Consider $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

(a) Using that the columns of A are orthogonal, find A^{-1}.

(b) Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_4$ be the columns of A. Without solving equations, find coefficients $c_1, \ldots, c_4$ such that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = c_1 \boldsymbol{w}_1 + \ldots + c_4 \boldsymbol{w}_4$.

Solution.

(a) Since the columns of A are orthogonal, we have: ($A^T A$ is diagonal and entries on the main diagonal are lengths of the columns of A)

$$A^T A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 2I$$

Hence, $A^{-1} = \frac{1}{2} A^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

(b) We have to find $c_1, \dots, c_4$ so that $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$. Hence, we have:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{7}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Short Problem 5. Let A be a $n \times n$ matrix with $A^T = A^{-1}$. What can you say about $\det(A)$?

Solution. We have:

$$1 = \det(I) = \det(A^{-1}A) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2$$

Hence, $\det(A) = 1$ or, -1.

Short Problem 6. Let A be an $n \times n$ matrix with eigenvalue λ. Determine whether each of the following statements is correct.

(a) λ^2 is an eigenvalue of A^2.
(b) λ^{-1} is an eigenvalue of A^{-1}.
(c) $\lambda + 1$ is an eigenvalue of $A + I$.
(d) λ cannot be zero.

Solution.

(a) True, let v be an eigenvector of A corresponding to λ. We have:

$$A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$$

Hence, λ^2 is an eigenvalue of A^2.

(b) True (if A is invertible), let v be an eigenvector of A corresponding to λ. We have:

$$Av = \lambda v \Rightarrow v = \lambda A^{-1} v \Rightarrow A^{-1} v = \lambda^{-1} v$$

Hence, λ^{-1} is an eigenvalue of A^{-1}.

(c) True, let v be an eigenvector of A corresponding to λ. We have:

$$(A + I)v = Av + v = \lambda v + v = (\lambda + 1)v$$

Hence, $\lambda + 1$ is an eigenvalue of $A + I$.

(d) False, consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Short Problem 7. Consider a matrix

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \star \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \star \\ \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} & \star \end{bmatrix},$$

in which the third column has not been specified, yet. Which of the following vectors can be added as a third column of Q such that Q is orthogonal?

(a) $\begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$,

(b) $\begin{bmatrix} \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{42}} \end{bmatrix}$,

(c) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

(d) none of the above.

Solution. The vector should have length 1 and it should be orthogonal to the first two columns. Hence, (b) is the right answer.

Short Problem 8. True or false?

(a) If A^TA is diagonal, then A has orthogonal columns.
(b) If A is an orthogonal matrix, then A^T is an orthogonal matrix.
(c) If $A\boldsymbol{x} = 0$, then $\boldsymbol{x}$ is orthogonal to the columns of A.
(d) For all $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.
(e) For all $n \times n$ matrices A and B, $\det(A + B) = \det(A) + \det(B)$.
(f) Every orthonormal set of vectors is linearly independent.
(g) Every subspace of $\mathbb{R}^n$ has an orthogonal basis.
(h) If every row of A adds up to 0, then $\det(A) = 0$.
(i) If every row of A adds up to 1, then $\det(A) = 1$.
(j) If A is invertible and B is not invertible, then AB is invertible.
(k) The determinant of A is the product of the diagonal entries of A.

Solution.

(a) True, note that the element in the ith row and jth column of A^TA is the dot product of the ith column and the jth column of A.
(b) True, since $A^TA = AA^T = I$ so A^T is also orthogonal. (Note: $(A^T)^TA^T = AA^T = I$)
(c) False, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. In fact, we have that $\boldsymbol{x}$ is orthogonal to the rows of A.
(d) True.
(e) False, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
(f) True, since if $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n = 0$ then $0 = \boldsymbol{v} \cdot \boldsymbol{v}_i = c_i$. Hence, all coefficients are equal to zero, i.e., they are linearly independent.
(g) True, every subspace of $\mathbb{R}^n$ has a basis and we can use Gram-Schmidt to find an orthonormal basis.
(h) True, the condition gives us that $A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0$, thus $A\boldsymbol{x} = 0$ has a nonzero solution, i.e., A is not invertible. Hence, $\det(A) = 0$.
(i) False, consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
(j) False, AB is invertible if and only if both A and B are invertible.
(k) False, consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Short Problem 9. Suppose that the projection matrix corresponding to orthogonal projection onto V is $P = \frac{1}{30}\begin{bmatrix} 29 & -2 & 5 \\ -2 & 26 & 10 \\ 5 & 10 & 5 \end{bmatrix}$.

(a) Is $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ in V?

(b) Find the vector in V which is closest to $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(c) What is the dimension of V?

Solution.

(a) We note that $\boldsymbol{v}$ is in V if and only if $P\boldsymbol{v} = \boldsymbol{v}$. Since

$$P\boldsymbol{v} = \frac{1}{30} \begin{bmatrix} 29 & -2 & 5 \\ -2 & 26 & 10 \\ 5 & 10 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \boldsymbol{v},$$

we conclude that $\boldsymbol{v}$ is in V.

(b) The vector in V which is closest to $\boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the orthogonal projection of $\boldsymbol{w}$ onto V. Since we have the projection matrix, this projection is

$$P\boldsymbol{w} = \frac{1}{30} \begin{bmatrix} 29 & -2 & 5 \\ -2 & 26 & 10 \\ 5 & 10 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 9 \\ 8 \\ 5 \end{bmatrix}.$$

(c) First, note that $V = \operatorname{Col}(P)$ (because the columns of P are the projections of the standard basis vectors onto V). Clearly, $\dim \operatorname{Col}(P) \geq 2$ (because the columns are not just multiples of each other). Hence, $\dim V \geq 2$. On the other hand, V cannot be three-dimensional, because we just saw that $\boldsymbol{w}$ is not in V. We conclude that $\dim V = 2$.