The geometry of linear equations

Adding and scaling vectors

Example 1. We have already encountered **matrices** such as

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & -1 & 2 & 2 \\ 3 & 2 & -2 & 0 \end{bmatrix}.$$

Each column is what we call a **(column) vector**.

In this example, each column vector has 3 entries and so lies in $\mathbb{R}^3$.

Example 2. A fundamental property of vectors is that vectors of the same kind can be **added** and **scaled**.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \quad , \qquad 7 \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \quad .$$

Example 3. (Geometric description of $\mathbb{R}^2$) A vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ represents the point (x_1, x_2) in the plane.

Given $\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, graph $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{x} + \boldsymbol{y}$, $3\boldsymbol{x}$.

Adding and scaling vectors, the most general thing we can do is:

Definition 4. Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_m$, the vector

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m$$

is a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

The scalars $c_1, \ldots, c_m$ are the **coefficients** or **weights**.

Example 5. Linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ include:

- $3\boldsymbol{v}_1 - \boldsymbol{v}_2 + 7\boldsymbol{v}_3$,
- $\boldsymbol{v}_2 + \boldsymbol{v}_3$,
- $\frac{1}{3}\boldsymbol{v}_2$,
- 0.

Example 6. Express $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Solution. We have to find c_1 and c_2 such that ...

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Example 7. We can think of the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

in two different geometric ways.

Row picture.

Each equation defines a line in $\mathbb{R}^2$.

Which points lie on the intersection of these lines?

Column picture.

The system can be written as $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Which linear combinations of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ produce $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$?

This example has the unique solution $x = 2$, $y = 3$.

- $(2, 3)$ is the (only) intersection of the two lines $2x - y = 1$ and $x + y = 5$.
- $2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the (only) linear combination producing $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Example 8. Consider the vectors

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}.$$

Determine if $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

Solution. Vector $\boldsymbol{b}$ is a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ if we can find weights x_1, x_2, x_3 such that:

This vector equation corresponds to the linear system:

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Corresponding augmented matrix:

Row reduction to echelon form:

Hence:

Example 9. In the previous example, express $\boldsymbol{b}$ as a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$.

Solution.

Summary

A vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \ldots + x_m\boldsymbol{a}_m = \boldsymbol{b}$$

has the same solution set as the linear system with augmented matrix

$$\left[\begin{array}{ccccc} | & | & & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m & \boldsymbol{b} \\ | & | & & | & | \end{array}\right].$$

In particular, $\boldsymbol{b}$ can be generated by a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ if and only if there is a solution to this linear system.

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The span of a set of vectors

Definition 10. The **span** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ is the set of all their linear combinations. We denote it by $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

In other words, $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ is the set of all vectors of the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m,$$

where $c_1, c_2, \ldots, c_m$ are scalars.

Example 11.

(a) Describe $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ geometrically.

(b) Describe $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$ geometrically.

(c) Describe $\operatorname{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right\}$ geometrically.

A single (nonzero) vector always spans a line, two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ usually span a plane but it could also be just a line (if $\boldsymbol{v}_2 = \alpha\boldsymbol{v}_1$).

We will come back to this when we discuss dimension and linear independence.

Example 12. Is $\operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}\right\}$ a line or a plane?

Solution.

Example 13. Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}.$$

Is $\boldsymbol{b}$ in the plane spanned by the columns of A?

Solution.

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Conclusion and summary

- The **span** of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m$ is the set of all their **linear combinations**.
- Some vector $\boldsymbol{b}$ is in $\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}$ if and only if there is a solution to the linear system with augmented matrix

$$\begin{bmatrix} | & | & & | & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m & \boldsymbol{b} \\ | & | & & | & | \end{bmatrix}.$$

  - Each solution corresponds to the weights in a linear combination of the $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots,$ $\boldsymbol{a}_m$ which gives $\boldsymbol{b}$.
  - This gives a second geometric way to think of linear systems!

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