The inverse of a matrix

Example 1. The inverse of a real number a is denoted as a^{-1}. For instance, $7^{-1} = \frac{1}{7}$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1.$$

In the context of $n \times n$ matrix multiplication, the role of 1 is taken by the $n \times n$ identity matrix

$$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Definition 2. An $n \times n$ matrix A is **invertible** if there is a matrix B such that

$$AB = BA = I_n.$$

In that case, B is the **inverse** of A and we write $A^{-1} = B$.

Example 3. We already saw that elementary matrices are **invertible**.

- $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} =$

Note.

- The inverse of a matrix is unique. Why? — So A^{-1} is well-defined.
- Do not write $\frac{A}{B}$. Why?
- If $AB = I$, then $BA = I$ (and so $A^{-1} = B$). — Not easy to show at this stage.

Example 4. The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible. Why?

Solution.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

Example 5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Armin Straub
astraub@illinois.edu

Let's check that:

Note.

- A 1×1 matrix $\left[\begin{array}{c} a \end{array}\right]$ is invertible $\Longleftrightarrow$ $a \neq 0$.
- A 2×2 matrix $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ is invertible $\Longleftrightarrow$ $ad - bc \neq 0$.

We will encounter the quantities on the right again when we discuss determinants.

Theorem 6. Suppose A and B are invertible. Then:

- A^{-1} is invertible and $(A^{-1})^{-1} = A$.

  Why?
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

  Why?

Solving systems using matrix inverse

Theorem 7. Let A be invertible. Then the system $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Proof. □

Example 8. Solve $\begin{array}{rrcl} -7x_1 & +3x_2 & = & 2 \\ 5x_1 & -2x_2 & = & 1 \end{array}$ using matrix inversion.

Solution. In matrix form $A\boldsymbol{x} = \boldsymbol{b}$, this system is

Computing the inverse:

Hence, the solution is:

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} =$$

Recipe for computing the inverse

To solve $A\boldsymbol{x} = \boldsymbol{b}$, we do row reduction on $\left[\begin{array}{c|c} A & \boldsymbol{b} \end{array}\right]$.

To solve $AX = I$, we do row reduction on $\left[\begin{array}{c|c} A & I \end{array}\right]$.

To compute A^{-1}:

- Form the augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.
- Compute the reduced echelon form.
- If A is invertible, the result is of the form $\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$.

Example 9. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution. By row reduction:

$$\left[\begin{array}{cc} A & I \end{array}\right] \rightsquigarrow \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$$

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

Example 10. Let's do the previous example step by step.

Note. Here is another way to see why this algorithm works:

- Each row reduction corresponds to multiplying with an elementary matrix E:

$$\left[\begin{array}{c|c} A & I \end{array}\right] \rightsquigarrow \left[\begin{array}{c|c} E_1 A & E_1 I \end{array}\right] \rightsquigarrow \left[\begin{array}{c|c} E_2 E_1 A & E_2 E_1 \end{array}\right] \rightsquigarrow \ldots$$

- So at each step:

$$\left[\begin{array}{c|c} A & I \end{array}\right] \rightsquigarrow \left[\begin{array}{c|c} FA & F \end{array}\right] \qquad \text{with } F = E_r \cdots E_2 E_1$$

- If we manage to reduce $\left[\begin{array}{c|c} A & I \end{array}\right]$ to $\left[\begin{array}{c|c} I & F \end{array}\right]$, this means

$$FA = I \qquad \text{and hence } A^{-1} = F.$$

Armin Straub
astraub@illinois.edu

Conclusions

Theorem 11. Let A be an $n \times n$ matrix. Then the following statements are equivalent: (i.e., for a given A, they are either all true or all false)

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivots. (Easy to check!)

(d) For every $\boldsymbol{b}$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

Namely, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

(e) There is a matrix B such that $AB = I_n$. (A has a "right inverse".)

(f) There is a matrix C such that $CA = I_n$. (A has a "left inverse".)

Note. Matrices that are not invertible are often called **singular**.

The book uses **singular** for $n \times n$ matrices that do not have n pivots. As we just saw, it doesn't make a difference.

Example 12. We now see at once that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Why?

Armin Straub
astraub@illinois.edu