Solving $Ax = 0$ and $Ax = b$

Null spaces

Definition 1. The **null space** of a matrix A is

$$\operatorname{Nul}(A) = \{\boldsymbol{x} : A\boldsymbol{x} = 0\}.$$

In other words, if A is $m \times n$, then its null space consists of those vectors $\boldsymbol{x} \in \mathbb{R}^n$ which solve the **homogeneous** equation $A\boldsymbol{x} = 0$.

Theorem 2. If A is $m \times n$, then $\operatorname{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Proof. We need to show that $\operatorname{Nul}(A)$ is closed under addition and scalar multiplication:

□

Solving $A\boldsymbol{x} = \mathbf{0}$ yields an *explicit description* of $\operatorname{Nul}(A)$.

By that we mean a description as the span of some vectors.

Example 3. Find an explicit description of $\operatorname{Nul}(A)$ where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

Solution.

Note. The number of vectors in the spanning set for $\operatorname{Nul}(A)$ as derived above (which is as small as possible) equals the number of free variables in $A\boldsymbol{x} = \mathbf{0}$.

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Column spaces

Definition 4. The **column space** $\mathrm{Col}(A)$ of a matrix A is the span of the columns of A.

If $A = [\ \boldsymbol{a}_1\ \ \ldots\ \ \boldsymbol{a}_n\ ]$, then $\mathrm{Col}(A) = \mathrm{span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$.

- In other words, $\boldsymbol{b}$ is in $\mathrm{Col}(A)$ if and only if $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

  Why?

- If A is $m \times n$, then $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

  Why?

Example 5. Find a matrix A such that $W = \mathrm{Col}(A)$ where

$$W = \left\{ \begin{bmatrix} 2x - y \\ 3y \\ 7x + y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}.$$

Solution.

$\mathrm{Col}(A)$ and solutions to $A\boldsymbol{x} = \boldsymbol{b}$

Theorem 6. Let $\boldsymbol{x}_p$ be a solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$.

Then every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is a solution to the homogeneous equation $A\boldsymbol{x} = \boldsymbol{0}$.

- In other words, $\{\boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}\} = \boldsymbol{x}_p + \mathrm{Nul}(A)$.
- We often call $\boldsymbol{x}_p$ a **particular solution**.

  The theorem then says that every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is the sum of a fixed chosen particular solution and some solution to $A\boldsymbol{x} = \boldsymbol{0}$.

Proof. Let $\boldsymbol{x}$ be another solution to $A\boldsymbol{x} = \boldsymbol{b}$. Then:

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□

Example 7. Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$.

Using the RREF, find a parametric description of the solutions to $A\boldsymbol{x} = \boldsymbol{b}$:

Every solution to $A\boldsymbol{x} = \boldsymbol{b}$ is therefore of the form:

Note. A convenient way to just find a particular solution is to set all free variables to zero.

Practice problems

- True or false?
  - The solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$ form a vector space.
  - The solutions to the equation $A\boldsymbol{x} = \boldsymbol{0}$ form a vector space.

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Example 8. Find an explicit description of $\mathrm{Nul}(A)$ where

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}.$$

Example 9. Is the given set W a vector space?

If possible, express W as the column or null space of some matrix A.

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x = y + 2z \right\}$

(b) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 5x - 1 = y + 2z \right\}$

(c) $W = \left\{ \begin{bmatrix} x \\ y \\ x + y \end{bmatrix} : x, y \text{ in } \mathbb{R} \right\}$

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