Bases for column and null spaces

Bases for null spaces

To find a basis for $\mathrm{Nul}(A)$:

- find the parametric form of the solutions to $A\boldsymbol{x} = \mathbf{0}$,
- express solutions $\boldsymbol{x}$ as a linear combination of vectors with the free variables as coefficients;
- these vectors form a basis of $\mathrm{Nul}(A)$.

Example 1. Find a basis for $\mathrm{Nul}(A)$ with

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}.$$

Solution.

Bases for column spaces

Recall that the columns of A are independent

$\iff$ $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution (namely, $\boldsymbol{x} = 0$),

$\iff$ A has no free variables.

A basis for $\mathrm{Col}(A)$ is given by the pivot columns of A.

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Example 2. Find a basis for $\mathrm{Col}(A)$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution.

Warning: For the basis of $\mathrm{Col}(A)$, you have to take the columns of A, not the columns of an echelon form.

Dimension of $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$

Theorem 3. Let A be an $m \times n$ matrix. Then:

- $\dim \mathrm{Col}(A)$ is the number of pivots of A
- $\dim \mathrm{Nul}(A)$ is the number of free variables of A
- $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = \ldots$

Proof.

□

Example 4. Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find the dimension of $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$.

Solution.

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The four fundamental subspaces

Row space and left null space

Definition 5.

- The **row space** of A is the column space of A^T.

- The **left null space** of A is the null space of A^T.

Definition 6. The **rank** of a A is the number of its pivots.

Theorem 7. (Fundamental Theorem of Linear Algebra, Part I)
Let A be an $m \times n$ matrix of rank r.

- $\dim \operatorname{Col}(A) = r$
- $\dim \operatorname{Col}(A^T) = r$
- $\dim \operatorname{Nul}(A) = n - r$ (number of free variables of A)
- $\dim \operatorname{Nul}(A^T) = m - r$

Example 8. Observe that the column space and the row space have the same dimension! Easy to see for a matrix in echelon form

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

but not obvious for a random matrix.

In particular, an $n \times n$ matrix has independent columns if and only if it has independent rows.

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Theorem 9. (Existence of inverses)

Let A be an $n \times n$ matrix. Then the following statements are equivalent:

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has rank n. (that is, A has n pivots)

(d) The columns of A span $\mathbb{R}^n$.

In other words, for every $\boldsymbol{b}$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution.

(e) The columns of A are independent.

In other words, the system $A\boldsymbol{x} = \boldsymbol{0}$ has only the solution $\boldsymbol{x} = \boldsymbol{0}$.

(f) For every $\boldsymbol{b}$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

Practice problems

Example 10. Suppose A is a 5×5 matrix, and that $\boldsymbol{v}$ is a vector in $\mathbb{R}^5$ which is not a linear combination of the columns of A.

What can you say about the number of solutions to $A\boldsymbol{x} = \boldsymbol{0}$?

Solution.

True or false?

- An $n \times n$ matrix A is invertible if and only if $\mathrm{Nul}(A) = \{\boldsymbol{0}\}$.
- An $n \times n$ matrix A is invertible if and only if the rows of A span $\mathbb{R}^n$.
- An $n \times n$ matrix A is invertible if and only if the rows of A are independent.

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