Linear transformations

Throughout, V and W are vector spaces.

Definition 1. A map $T: V \to W$ is a **linear transformation** if

$$T(c\boldsymbol{x} + d\boldsymbol{y}) = cT(\boldsymbol{x}) + dT(\boldsymbol{y}) \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \text{ in } V \text{ and all } c, d \text{ in } \mathbb{R}.$$

Example 2. Let A be an $m \times n$ matrix.

Then the map $T(\boldsymbol{x}) = A\boldsymbol{x}$ is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$.

Why?

Example 3. Let $\mathbb{P}_n$ be the vector space of all polynomials of degree at most n. Consider the map $T: \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by

$$T(p(t)) = \frac{\mathrm{d}}{\mathrm{d}t} p(t).$$

This map is linear! Why?

Representing linear maps by matrices

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for V.

A linear map $T: V \to W$ is determined by the values $T(\boldsymbol{x}_1), \ldots, T(\boldsymbol{x}_n)$.

Why?

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Definition 4. (From linear maps to matrices)

Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be a basis for V, and $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ a basis for W.

The **matrix representing** T with respect to these bases

- has n columns (one for each of the $\boldsymbol{x}_j$),
- the j-th column has m entries $a_{1,j}, \ldots, a_{m,j}$ determined by

$$T(\boldsymbol{x}_j) = a_{1,j}\boldsymbol{y}_1 + \ldots + a_{m,j}\boldsymbol{y}_m.$$

Example 5. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}.$$

What is the matrix $A(T)$ representing T with respect to the standard bases?

Solution.

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Example 6. As in the previous example, let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Let T be the linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}.$$

What is the matrix $B(T)$ representing T with respect to the following bases?

$$\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{x}_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\boldsymbol{x}_2} \text{ for } \mathbb{R}^2, \qquad \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\boldsymbol{y}_2}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{y}_3} \text{ for } \mathbb{R}^3.$$

Solution.

A matrix representing T encodes in column j the coefficients of $T(\boldsymbol{x}_j)$ expressed as a linear combination of $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$.

Example 7. Let $T: \mathbb{P}_3 \rightarrow \mathbb{P}_2$ be the linear map given by

$$T(p(t)) = \frac{\mathrm{d}}{\mathrm{d}t} p(t).$$

What is the matrix $A(T)$ representing T with respect to the standard bases?

Solution.

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Important geometric examples

We consider some linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ and their geometric interpretation.

Example 8. The matrix $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ ...

Example 9. The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ...

Example 10. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ...

Example 11. The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ...

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Example 12. The matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$...

Example 13. Let T be the linear map which projects each vector onto the line with slope θ.

- Which matrix represents T (with respect to the standard basis)?
- Give a basis of $\mathbb{R}^2$ with respect to which T is represented by a very simple matrix.

Solution.