Orthogonality

The inner product and distances

Definition 1. The **inner product** (or **dot product**) of $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbb{R}^n$:

$$\boldsymbol{v} \cdot \boldsymbol{w} \;=\; \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + \ldots + v_n w_n.$$

Example 2. For instance,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} =$$

Definition 3.

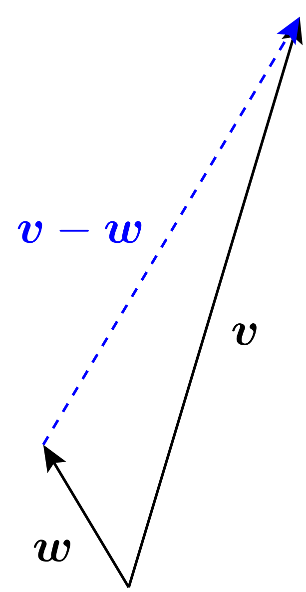
- The **norm** (or **length**) of a vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is

$$\|\boldsymbol{v}\| \;=\; \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + \ldots + v_n^2}.$$

This is the distance to the origin.

- The **distance** between points $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ is

$$\operatorname{dist}(\boldsymbol{v}, \boldsymbol{w}) \;=\; \|\boldsymbol{v} - \boldsymbol{w}\|.$$



Example 4. For instance, in $\mathbb{R}^2$,

$$\operatorname{dist}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) =$$

Orthogonal vectors

Definition 5. $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$ are **orthogonal** if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$

Armin Straub
astraub@illinois.edu

How is this related to our understanding of right angles?

Example 6. Are the following vectors orthogonal?

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Theorem 7. Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are nonzero and pairwise orthogonal. Then $\boldsymbol{v}_1, \ldots,$ $\boldsymbol{v}_n$ are independent.

Proof. Suppose that

$$c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n = \boldsymbol{0}.$$

□

Example 8. Let us consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$.

Find $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$. Observe!

Solution.

Example 9. Repeat for $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

Solution.

The fundamental theorem, second act

Definition 10. Let W be a subspace of $\mathbb{R}^n$, and $\boldsymbol{v}$ in $\mathbb{R}^n$.

- $\boldsymbol{v}$ is **orthogonal** to W, if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ for all $\boldsymbol{w}$ in W.
- Another subspace V is **orthogonal** to W, if every vector in V is orthogonal to W.
- The **orthogonal complement** of W is the space $W^\perp$ of all vectors that are orthogonal to W.

  Exercise: show that the orthogonal complement is indeed a vector space.

Example 11. In the previous example, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$.

We found that

$$\operatorname{Nul}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \operatorname{Col}(A^T) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

are orthogonal subspaces.

Theorem 12. (Fundamental Theorem of Linear Algebra, Part I)

Let A be an $m \times n$ matrix of rank r.

- $\dim \operatorname{Col}(A) = r$ (subspace of $\mathbb{R}^m$)
- $\dim \operatorname{Col}(A^T) = r$ (subspace of $\mathbb{R}^n$)
- $\dim \operatorname{Nul}(A) = n - r$ (subspace of $\mathbb{R}^n$)
- $\dim \operatorname{Nul}(A^T) = m - r$ (subspace of $\mathbb{R}^m$)

Armin Straub
astraub@illinois.edu

Theorem 13. **(Fundamental Theorem of Linear Algebra, Part II)**

- $\mathrm{Nul}(A)$ is orthogonal to $\mathrm{Col}(A^T)$. (both subspaces of $\mathbb{R}^n$)

  Note that $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A^T) = n$.

  Hence, the two spaces are orthogonal complements.

- $\mathrm{Nul}(A^T)$ is orthogonal to $\mathrm{Col}(A)$.

  Again, the two spaces are orthogonal complements.

Why?

Corollary 14. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable
$\Longleftrightarrow$ $\boldsymbol{y}^T\boldsymbol{b} = 0$ whenever $\boldsymbol{y}^T A = \boldsymbol{0}$

Proof.

□

Motivation

Example 15. Not all linear systems have solutions.

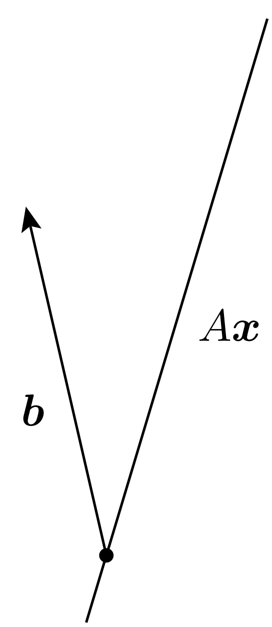
In fact, for many applications, data needs to be fitted and there is no hope for a perfect match.

For instance, $A\boldsymbol{x} = \boldsymbol{b}$ with

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

has no solution:

- $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is not in $\mathrm{Col}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$
- Instead of giving up, we want the $\boldsymbol{x}$ which makes $A\boldsymbol{x}$ and $\boldsymbol{b}$ as close as possible.
- Such $\boldsymbol{x}$ is characterized by ...



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astraub@illinois.edu