Orthogonal bases

- Recall: Suppose that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are nonzero and (pairwise) orthogonal. Then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are independent.

Definition 1. A basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of a vector space V is an **orthogonal basis** if the vectors are (pairwise) orthogonal.

Example 2. Are the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ an orthogonal basis for $\mathbb{R}^3$?

Solution.

Note that we do not need to check that the three vectors are independent. That follows from their orthogonality.

Example 3. Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and that $\boldsymbol{w}$ is in V. Find $c_1, \ldots, c_n$ such that

$$\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n.$$

Solution. Take the dot product of $\boldsymbol{v}_1$ with both sides:

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1\boldsymbol{v}_1 + \ldots + c_n\boldsymbol{v}_n \qquad \text{with} \quad c_j = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_j}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}.$$

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Example 4. Express $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solution.

Definition 5. A basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of a vector space V is an **orthonormal basis** if the vectors are orthogonal and have length 1.

If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthonormal basis of V, and $\boldsymbol{w}$ is in V, then

$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \qquad \text{with} \quad c_j = \boldsymbol{v}_j \cdot \boldsymbol{w}.$$

Example 6. Is the basis $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ orthonormal? If not, normalize the vectors to produce an orthonormal basis.

Solution.

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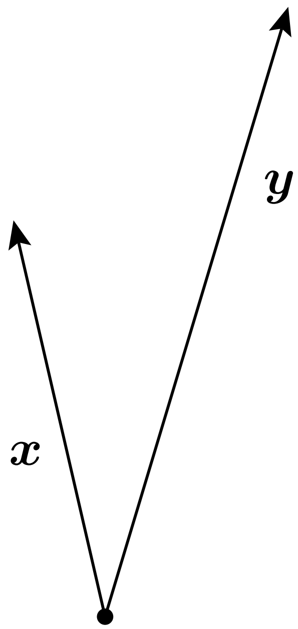
Orthogonal projections



Definition 7. The **orthogonal projection** of vector $\boldsymbol{x}$ onto vector $\boldsymbol{y}$ is

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}} \boldsymbol{y}.$$

- The vector $\hat{\boldsymbol{x}}$ is the closest vector to $\boldsymbol{x}$, which is in $\operatorname{span}\{\boldsymbol{y}\}$.
- The "error" $\boldsymbol{x}^{\perp} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ is orthogonal to $\operatorname{span}\{\boldsymbol{y}\}$.

$\boldsymbol{x}^{\perp} = \boldsymbol{x} - \hat{\boldsymbol{x}} = \boldsymbol{x} - \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\boldsymbol{y} \cdot \boldsymbol{y}} \boldsymbol{y}$ is also referred to as the **component of $\boldsymbol{x}$ orthogonal to $\boldsymbol{y}$**.

Example 8. What is the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ onto $\boldsymbol{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

Solution.

Example 9. What are the orthogonal projections of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto each of the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

Solution.

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Recall: If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ is an orthogonal basis of V, and $\boldsymbol{w}$ is in V, then

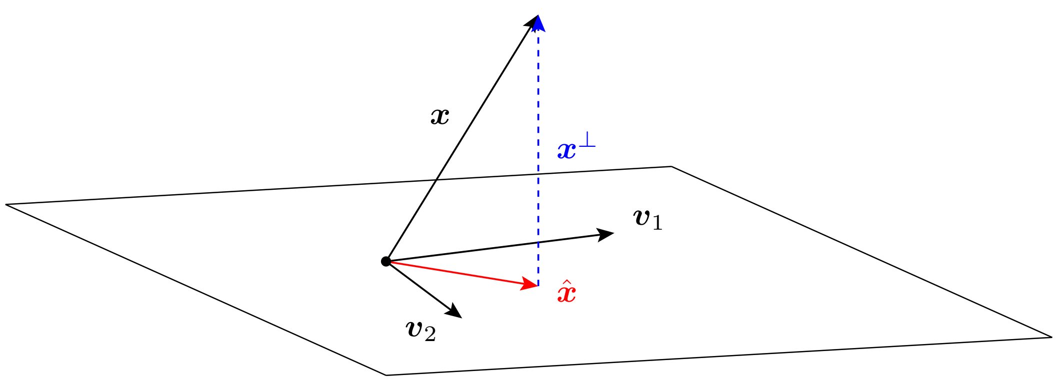
$$\boldsymbol{w} = c_1 \boldsymbol{v}_1 + \ldots + c_n \boldsymbol{v}_n \qquad \text{with} \quad c_j = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_j}{\boldsymbol{v}_j \cdot \boldsymbol{v}_j}.$$

$\rightsquigarrow$ $\boldsymbol{w}$ decomposes as the sum of its projections onto each basis vector

Orthogonal projection on subspaces

Theorem 10. Let W be a subspace of $\mathbb{R}^n$. Then, each $\boldsymbol{x}$ in $\mathbb{R}^n$ can be uniquely written as

$$\boldsymbol{x} = \underbrace{\hat{\boldsymbol{x}}}_{\text{in } W} + \underbrace{\boldsymbol{x}^{\perp}}_{\text{in } W^{\perp}}.$$



- $\hat{\boldsymbol{x}}$ is the **orthogonal projection** of $\boldsymbol{x}$ onto W.
- $\hat{\boldsymbol{x}}$ is the point in W closest to $\boldsymbol{x}$. For any other $\boldsymbol{y}$ in W,

$$\text{dist}(\boldsymbol{x}, \hat{\boldsymbol{x}}) < \text{dist}(\boldsymbol{x}, \boldsymbol{y}).$$

- If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ is an orthogonal basis of W, then

Example 11. Let $W = \text{span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, and $\boldsymbol{x} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$.

- Find the orthogonal projection of $\boldsymbol{x}$ onto W.
- Write $\boldsymbol{x}$ as a vector in W plus a vector orthogonal to W.

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Solution.

Theorem 12. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ be an orthogonal basis of W, a subspace of $\mathbb{R}^n$. The projection map $\pi_W: \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\boldsymbol{x} \mapsto \left(\frac{\boldsymbol{x} \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\right)\boldsymbol{v}_1 + \ldots + \left(\frac{\boldsymbol{x} \cdot \boldsymbol{v}_m}{\boldsymbol{v}_m \cdot \boldsymbol{v}_m}\right)\boldsymbol{v}_m$$

is linear. The matrix P representing π_W with respect to the standard basis is the corresponding **projection matrix**.

Example 13. Find the projection matrix P which corresponds to orthogonal projection onto $W = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

Solution.

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Example 14. Compute P^2 for the projection matrix we just computed. Explain!

Solution.

Practice problems

Example 15. Find the closest point to $\boldsymbol{x}$ in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, where

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. This is the orthogonal projection of $\boldsymbol{x}$ onto $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

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