Gram–Schmidt

Recipe: (Gram–Schmidt orthonormalization)

Given a basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, produce an orthonormal basis $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$.

$$\begin{aligned} \boldsymbol{b}_1 &= \boldsymbol{a}_1, & \boldsymbol{q}_1 &= \frac{\boldsymbol{b}_1}{\|\boldsymbol{b}_1\|} \\ \boldsymbol{b}_2 &= \boldsymbol{a}_2 - \langle \boldsymbol{a}_2, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1, & \boldsymbol{q}_2 &= \frac{\boldsymbol{b}_2}{\|\boldsymbol{b}_2\|} \\ &\vdots \end{aligned}$$

Example 2. Find an orthonormal basis for $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution.

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Definition 3. An **orthogonal matrix** is a square matrix with orthonormal columns.

Theorem 4. An $n \times n$ matrix Q is orthogonal $\iff$ $Q^T Q = I$

Proof.

$\square$

Example 5. $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Q is orthogonal because

The QR decomposition (flashed at you)

Let A be an $m \times n$ matrix of rank n. (columns independent)

Then we have the **QR decomposition** $A = QR$,

- where Q has orthonormal columns, and
- R is upper triangular and invertible.

Idea: Gram–Schmidt on the columns of A, to get the columns of Q!

Example 6. Find the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.

Solution.

In general, $A = QR$ is obtained as:

$$\begin{bmatrix} | & | & \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots \\ | & | & \end{bmatrix} = \begin{bmatrix} | & | & \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} \langle \boldsymbol{a}_1, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{a}_2, \boldsymbol{q}_1 \rangle & \langle \boldsymbol{a}_3, \boldsymbol{q}_1 \rangle & \cdots \\ & \langle \boldsymbol{a}_2, \boldsymbol{q}_2 \rangle & \langle \boldsymbol{a}_3, \boldsymbol{q}_2 \rangle & \\ & & \langle \boldsymbol{a}_3, \boldsymbol{q}_3 \rangle & \\ & & & \ddots \end{bmatrix}$$

Practice problems

Example 7. Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

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