Determinants

Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} =$$

The **determinant** of

- a 2×2 matrix is $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) =$
- a 1×1 matrix is $\det([\,a\,]) =$

Goal: A is invertible $\iff$ $\det(A) \neq 0$

We will write both $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for the determinant.

Definition 1. The **determinant** is characterized by:

- the normalization $\det I = 1$,
- and how it is affected by elementary row operations:
  - **(replacement)** Add one row to a multiple of another row.
  - **(interchange)** Interchange two rows.
  - **(scaling)** Multiply all entries in a row by s.

Example 2. Compute $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{vmatrix}$.

Solution.

Example 3. Compute $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{vmatrix}$.

Solution.

The determinant of a triangular matrix is

Example 4. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$.

Solution. We do row reductions:

Example 5. Discover the formula for $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Solution.

Example 6. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$.

Solution.

The following important properties follow from the behaviour under row operations.

- $\det(A) = 0 \iff A$ is not invertible

  Why?

- $\det(AB) =$
- $\det(A^{-1}) =$
- $\det(A^T) =$

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Example 7. Recall that $AB = \mathbf{0}$, then it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$. However, show that $\det(A) = 0$ or $\det(B) = 0$.

Solution.

Example 8. Suppose A is a 3×3 matrix with $\det(A) = 5$. What is $\det(2A)$?

Solution.

A "bad" way to compute determinants

Example 9. Compute $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ by **cofactor expansion**.

Solution. We expand by the first row:

Each term in the cofactor expansion is ± 1 times an entry times a smaller determinant (row and column of entry deleted).

The ± 1 is assigned to each entry according to $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$.

Solution. We expand by the second column:

Solution. We expand by the third column:

Why is the method of cofactor expansion not practical?

Practice problems

Problem 1. Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution. The final answer should be -10.

Problem 2. Let A be an $n \times n$ matrix.

Express the following in terms of $\det(A)$:

- $\det(A^2) =$
- $\det(2A) =$

  Hint: (unless $n = 1$) this is not just $2\det(A)$

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