Preparing for the Final

MATH 237 — Linear Algebra I
Fall 2016

Please print your name:

Our final exam will be comprehensive, with a focus on the material learned later in the semester.

(Note that lots of the things we learned more recently require us to know earlier material anyway.)

A good way to prepare yourself is to study the following:

- redo the practice problems for Midterm 1 and Midterm 2,
- do the problems below,
- retake the midterm exams and quizzes,
- go through the lecture sketches.

  Make sure that you can briefly but precisely define our important notions (linear independence, basis, rank, dimension, ...). These are in bold face in the lecture sketches. The sketches also contain lots of (computationally pleasant) problems with solutions.

I Computational part

Problem 1. Let $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues and bases for the eigenspaces of A.

(b) If possible, diagonalize A. That is, determine matrices P and D such that $A = PDP^{-1}$.

Solution.

(a) By expanding by the third column, and then by the third row, we find that the characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 0 & 0 & 1 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 4-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix}$$

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$$= (4-\lambda)(1-\lambda)(2-\lambda)^2.$$

The eigenvalues are $\lambda = 1, 2, 2, 4$.

- For $\lambda = 1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} -1 \\ 1 \\ -1/3 \\ 1 \end{bmatrix}$.

  This follows from: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_2 - R_1 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{3}R_3 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- For $\lambda = 4$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -2 & 0 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

  This is obvious!

- For $\lambda = 2$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

  It is visible that $\operatorname{rank}\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right) = 3$, so that the 2-eigenspace only has dimension 1.

(b) The matrix A is not diagonalizable, because there are not enough linearly independent eigenvectors: the eigenvalue 2 has multiplicity 2 but the 2-eigenspace only has dimension 1.

□

Problem 2. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

(a) Find the eigenvalues and bases for the eigenspaces of A.

(b) If possible, diagonalize A. That is, determine matrices P and D such that $A = PDP^{-1}$.

Solution.

(a) By expanding by the second row, we find that the characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & -5-\lambda & 0 \\ 1 & 8 & 1-\lambda \end{vmatrix} = (-5-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (-5-\lambda)[(1-\lambda)^2 - 1] = (-5-\lambda)\lambda(\lambda-2).$$

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Hence, the eigenvalues are $\lambda = 0, 2, -5$.

- For $\lambda = 0$, the eigenspace $\text{null}\left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

- For $\lambda = 2$, the eigenspace $\text{null}\left(\begin{bmatrix} -1 & 2 & 1 \\ 0 & -7 & 0 \\ 1 & 8 & -1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

- For $\lambda = -5$, the eigenspace $\text{null}\left(\begin{bmatrix} 6 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 8 & 6 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 2/23 \\ -35/46 \\ 1 \end{bmatrix}$.

  This requires some work:

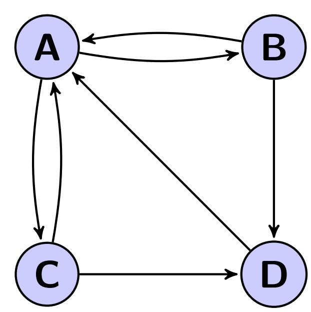
  $$\begin{bmatrix} 6 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 8 & 6 \end{bmatrix} \overset{R_3 - \frac{1}{6}R_1 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 6 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & \frac{23}{3} & \frac{35}{6} \end{bmatrix} \overset{R_2 \Leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 6 & 2 & 1 \\ 0 & \frac{23}{3} & \frac{35}{6} \\ 0 & 0 & 0 \end{bmatrix} \overset{\substack{\frac{1}{6}R_1 \Rightarrow R_1 \\ \frac{3}{23}R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{6} \\ 0 & 1 & \frac{35}{46} \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 - \frac{1}{3}R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -\frac{2}{23} \\ 0 & 1 & \frac{35}{46} \\ 0 & 0 & 0 \end{bmatrix}$$

(b) A possible choice is $P = \begin{bmatrix} -1 & 1 & 2/23 \\ 0 & 0 & -35/46 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.

□

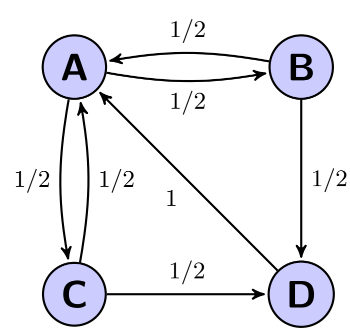
Problem 3. Suppose the internet consists of only the four webpages A, B, C, D which link to each other as indicated in the diagram.

Rank these webpages by computing their PageRank vector.



Solution. Recall that we model a random surfer, who randomly clicks on links. Let a_t be the probability that such a surfer will be on page A at time t. Likewise, b_t, c_t, d_t are the probabilities that the surfer will be on page B, C or D.

The transition probabilities are indicated in the diagram to the right.



$$\begin{bmatrix} a_{t+1} \\ b_{t+1} \\ c_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \cdot a_t + \frac{1}{2} \cdot b_t + \frac{1}{2} \cdot c_t + 1 \cdot d_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ \frac{1}{2} \cdot a_t + 0 \cdot b_t + 0 \cdot c_t + 0 \cdot d_t \\ 0 \cdot a_t + \frac{1}{2} \cdot b_t + \frac{1}{2} \cdot c_t + 0 \cdot d_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}}_{=T} \begin{bmatrix} a_t \\ b_t \\ c_t \\ d_t \end{bmatrix}$$

To find the equilibrium state, we determine an appropriate 1-eigenvector of the transition matrix T.

The 1-eigenspace is $\operatorname{null}(T - 1 \cdot I) = \operatorname{null}\left(\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}\right)$

To compute a basis, we perform Gaussian elimination (details below): $\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

We conclude that the 1-eigenspace has basis $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. (Note that its entries add up to $2+1+1+1=5$.)

The corresponding equilibrium state is $\frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$. This is the PageRank vector.

Correspondingly, we rank A the highest, followed by B, C, D which we rank equally.

[In hindsight, can you (at least sort of) see, directly from the diagram, why the PageRank is what it is?]

The full steps of the Gaussian elimination are:

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \overset{\substack{R_2 + \frac{1}{2}R_1 \Rightarrow R_2 \\ R_3 + \frac{1}{2}R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \overset{\substack{R_3 + \frac{1}{3}R_2 \Rightarrow R_3 \\ R_4 + \frac{2}{3}R_2 \Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \overset{R_4 + R_3 \Rightarrow R_4}{\rightsquigarrow} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{\substack{-1R_1 \Rightarrow R_1 \\ -\frac{4}{3}R_2 \Rightarrow R_2 \\ -\frac{3}{2}R_3 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\substack{R_1 + \frac{1}{2}R_3 \Rightarrow R_1 \\ R_2 + \frac{1}{3}R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 + \frac{1}{2}R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This was good practice of elimination! However, notice that we can actually find an eigenvector $\boldsymbol{x}$ with less effort by spelling out the equations: for instance, the second one is just $\frac{1}{2}x_1 - x_2 = 0$. Do that! □

Problem 4. Find a basis and the dimension of $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

Solution.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 \end{bmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_4 - 3R_1 \Rightarrow R_4}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \end{bmatrix} \overset{\substack{\text{permute} \\ \text{rows}}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_3 + 3R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{R_4 - \frac{1}{4}R_3 \Rightarrow R_4}{\rightsquigarrow} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Not a pivot in every column, hence the 4 vectors are dependent.

Moreover, a basis for W is $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\dim W = 3$. □

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II Short answer part

Problem 5. Suppose A is a 5×5 matrix with eigenvalue 0.

(a) What can you say about $\operatorname{rank}(A)$?

(b) What can you say about $\operatorname{rank}(A)$ if the multiplicity of 0 is 1?

(c) What can you say about $\operatorname{rank}(A)$ if the multiplicity of 0 is 2?

Solution.

(a) Note that the 0-eigenspace of A is just $\operatorname{null}(A)$. We therefore know that $\dim \operatorname{null}(A) \geqslant 1$. Equivalently, we know that $\operatorname{rank}(A) \leqslant 5 - 1 = 4$.

(b) If the multiplicity of 0 is 1, then $\dim \operatorname{null}(A) = 1$ and we know that $\operatorname{rank}(A) = 5 - 1 = 4$.

(c) If the multiplicity of 0 is 2, then $\dim \operatorname{null}(A) \in \{1, 2\}$ and we know that $\operatorname{rank}(A) \in \{3, 4\}$.

□

Problem 6. Produce a 2×2 matrix which has 1-eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and 3-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Are there others?

Solution. Because we have two independent eigenvectors, such a matrix A is diagonalizable as $A = PDP^{-1}$, and we know a possible choice of P and D, namely $P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Hence,

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -4 \\ -2 & 7 \end{bmatrix},$$

and this is the unique 2×2 matrix with 1-eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and 3-eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. □

Problem 7.

(a) What does it mean for two matrices A, B to be similar?

(b) Show that similar matrices have the same characteristic polynomial.

(c) Is it true that similar matrices have the same eigenvalues? Is it true that similar matrices have the same eigenvectors? Explain.

Solution.

(a) It means that there exists an invertible matrix P such that $A = PBP^{-1}$.

(b) Let A and B be similar. Then $A = PBP^{-1}$ for some invertible matrix P, and

$$\begin{aligned} \det(A - \lambda I) &= \det(PBP^{-1} - \lambda I) \\ &= \det(PBP^{-1} - P\lambda I P^{-1}) \\ &= \det(P(B - \lambda I)P^{-1}) \\ &= \det(P)\det(B - \lambda I)\det(P^{-1}) \\ &= \det(B - \lambda I). \end{aligned}$$

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In other words, A and B have the same characteristic polynomial.

(c) It is true that similar matrices have the same eigenvalues; that's because they have the same characteristic polynomials (and the eigenvalues are just the roots of that same polynomial). However, similar matrices do not typically have the same eigenvectors (think of any of the examples in which we diagonalized a matrix A as $A = PDP^{-1}$; the matrices A and D are similar but they have different eigenvectors). □

Problem 8. Let A be a $n \times n$ matrix. List at least five other statements which are equivalent to the statement "A is invertible".

Solution. Here are a few possibilities:

A is invertible.
$\Longleftrightarrow$ The RREF of A is I_n.
$\Longleftrightarrow$ A has n pivots.
$\Longleftrightarrow$ $\text{rank}(A) = 0$
$\Longleftrightarrow$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.
$\Longleftrightarrow$ The system $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution.
$\Longleftrightarrow$ $\dim \text{null}(A) = 0$
$\Longleftrightarrow$ The columns of A are linearly independent.
$\Longleftrightarrow$ The rows of A are linearly independent.
$\Longleftrightarrow$ For every $\boldsymbol{b} \in \mathbb{R}^n$, the system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.
$\Longleftrightarrow$ The columns of A span all of $\mathbb{R}^n$.
$\Longleftrightarrow$ $\dim \text{col}(A) = n$
$\Longleftrightarrow$ The rows of A span all of $\mathbb{R}^n$.
$\Longleftrightarrow$ $\dim \text{row}(A) = n$
$\Longleftrightarrow$ $\det(A) \neq 0$
$\Longleftrightarrow$ 0 is not an eigenvalue of A.

Make sure that you can explain why each statement is equivalent to A being invertible. □

Problem 9. Determine whether each of the following "laws" is true for all (invertible) $n \times n$ matrices A, B.

(a) $(AB)^T = A^T B^T$

(b) $(AB)^T = B^T A^T$

(c) $(AB)^{-1} = A^{-1} B^{-1}$

(d) $(AB)^{-1} = B^{-1} A^{-1}$

Solution.

(a) Not true for all A, B.

(b) True.

(c) Not true for all A, B.

(d) True. (Can you demonstrate why?) □

Problem 10. Describe $\text{col}(A), \text{row}(A), \text{null}(A)$ if A is an invertible $n \times n$ matrix.

Solution. Recall that A is invertible if and only if its RREF is I_n, the $n \times n$ identity matrix.

In particular, $\dim \text{col}(A) = n$, $\dim \text{row}(A) = n$, $\dim \text{null}(A) = 0$.

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Consequently, $\mathrm{col}(A) = \mathbb{R}^n$, $\mathrm{row}(A) = \mathbb{R}^n$, $\mathrm{null}(A) = \{\mathbf{0}\}$. □

Problem 11. You overhear a conversation during which someone explains that "matrix inverses are amazing because they allow us to solve any linear system $A\boldsymbol{x} = \boldsymbol{b}$ by simply computing $\boldsymbol{x} = A^{-1}\boldsymbol{b}$". What is your take on this statement?

Solution. It is true that, if A is invertible, then the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. There is however several reasons the fellow you overheard is a little too enthusiastic about inverses in that context:

- A is not always invertible, so this is not a general method. Think for instance of the equations we need to solve when finding eigenvectors: the corresponding matrices are never invertible!

- Also, recall that, in order to be invertible, the matrix A needs to be square. Thus we cannot use inverses in this way to solve any $m \times n$ system for which $m \neq n$.

- Thirdly, even if A is $n \times n$ and invertible, computing A^{-1} is more work than solving $A\boldsymbol{x} = \boldsymbol{b}$ (in our usual approach, we are doing the same steps of Gaussian elimination: for $A\boldsymbol{x} = \boldsymbol{b}$, the augmented matrix $[A \mid \boldsymbol{b}]$ has just one column besides A, whereas, for A^{-1}, the augmented matrix $[A \mid I]$ has n columns besides A.

  (However, if we need to solve $A\boldsymbol{x} = \boldsymbol{b}$ for many different right-hand sides $\boldsymbol{b}$, then computing A^{-1} will actually be more efficient.) □