Definition 15. An **elementary row operation** is one of the following:

- **(add)** Add a multiple of one row to another. For instance, $R_2 + 3R_1 \Rightarrow R_2$.
- **(scale)** Multiply a row by a nonzero constant. For instance, $\frac{1}{2}R_1 \Rightarrow R_1$.
- **(swap)** Interchange two rows. For instance, $R_1 \Leftrightarrow R_2$.

Two matrices are **row equivalent**, if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Theorem 16.

(a) If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution(s).

(b) **(Uniqueness of the reduced echelon form)** Each matrix is row equivalent to one and only one row-reduced echelon matrix (RREF).

4 Existence and uniqueness of solutions

After row reduction to echelon form, we can easily solve a linear system. At that stage, we can also tell (just by looking at it) how many solutions the system has (0, 1 or ∞).

Example 17. Find the general solution of the following linear system:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 3 \\ 2x_1 & + & x_2 & + & 4x_3 & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 1 \end{array}$$

Solution. We do Gaussian elimination to produce an echelon form:

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 2 & 1 & 4 & 0 \\ 1 & 2 & 2 & 1 \end{array}\right] \underset{\rightsquigarrow}{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 - R_1 \Rightarrow R_3}} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 0 & 3 & 0 & -6 \\ 0 & 3 & 0 & -2 \end{array}\right] \underset{\rightsquigarrow}{R_3 - R_2 \Rightarrow R_3} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 0 & 3 & 0 & -6 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Note that the last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 4$, or $0 = 4$, which is obviously false. This means that this system cannot have a solution. It is inconsistent.

Note. Once we put a system in echelon form, this is the only way in which it can be inconsistent. This is part of the existence and uniqueness theorem stated below.

Example 18. Find the general solution of the following linear system:

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & - & x_4 & = & 1 \\ 3x_1 & + & 6x_2 & - & 2x_3 & + & x_4 & = & 8 \end{array}$$

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Solution. A single operation produces an echelon form:

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & -1 & 1 \\ 3 & 6 & -2 & 1 & 8 \end{array}\right] \overset{R_2-3R_1\Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & 4 & 5 \end{array}\right]$$

- The pivots are located in columns $1, 3$.
  The corresponding variables x_1, x_3 are called **leading variables** (or **pivot variables**).
- The remaining variables x_2, x_4 are called **free variables**.

[We have no equations to solve for the free variables. Instead, the free variables can take any values.]

We set $x_2 = s_1$ and $x_4 = s_2$, where s_1, s_2 can be any numbers (free parameters).

Solving each equation for the pivot variable, we find (by back-substitution) that the **general solution** (in parametric form) of this system is:

$$\begin{cases} x_1 = 6 - 2s_1 - 3s_2 \\ x_2 = s_1 \\ x_3 = 5 - 4s_2 \\ x_4 = s_2 \end{cases}$$

[As mentioned, s_1 and s_2 can be any numbers. The resulting values for x_1, x_2, x_3, x_4 always solve the system. Our solution is general, meaning that there are no further solutions.]

Alternative. Alternatively, one more operation yields the RREF:

$$\overset{R_1+R_2\Rightarrow R_1}{\rightsquigarrow} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 5 \end{array}\right]$$

Again, we have the free variables $x_2 = s_1$ and $x_4 = s_2$. The general solution is as before, but this time, we can just read it off directly (no back-substitution).

Theorem 19. (Existence and uniqueness theorem) A linear system in echelon form is **inconsistent** if and only if it has a row of the form

$$[\; 0 \;\; 0 \;\; \ldots \;\; 0 \,|\, b \;],$$

where b is nonzero.

If a linear system is consistent, then the solutions consist of either

- a unique solution (when there are no free variables) or
- infinitely many solutions (when there is at least one free variable).

Example 20. For what values of h will the following system be consistent?

$$\begin{array}{rrcl} 3x_1 & -9x_2 & = & 4 \\ -2x_1 & +6x_2 & = & h \end{array}$$

Solution. We perform row reduction to find an echelon form:

$$\left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \overset{R_2+\frac{2}{3}R_1\Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{rr|r} 3 & -9 & 4 \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

Whether or not the system is consistent is determined by whether or not $h + \frac{8}{3} = 0$.
The system is consistent if and only if $h = -\frac{8}{3}$.

(In the [single] case in which it is consistent, it has infinitely many solutions [because x_2 is a free variable].)