**Example 37.**

- $\operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ consists of all multiples of $\begin{bmatrix}2\\1\end{bmatrix}$. Geometrically, this is a line.
- $\operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ consists of all vectors in $\mathbb{R}^2$. Geometrically, this is a plane.

Make a sketch to see why this is obvious. Also, we worked this out last class computationally.

- $\operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}-4\\-2\end{bmatrix}\right\}=\operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ is just a line again.

**Example 38.** $\operatorname{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}2\\3\\1\end{bmatrix}\right\}$ (from previous class) describes a plane.

**Solution.** Which vectors $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ are contained in that span?

$$\left[\begin{array}{cc|c}1&2&b_1\\1&3&b_2\\2&1&b_3\end{array}\right]\overset{\substack{R_2-R_1\Rightarrow R_2\\R_3-2R_1\Rightarrow R_3}}{\rightsquigarrow}\left[\begin{array}{cc|c}1&2&b_1\\0&1&b_2-b_1\\0&-3&b_3-2b_1\end{array}\right]\overset{R_3+3R_2\Rightarrow R_3}{\rightsquigarrow}\left[\begin{array}{cc|c}1&2&b_1\\0&1&b_2-b_1\\0&0&b_3+3b_2-5b_1\end{array}\right]$$

Hence, a vector $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ is in that span if and only if $-5b_1+3b_2+b_3=0$. This is the equation of a plane!

**Checking things.** Let's compare with last class: indeed, $\begin{bmatrix}0\\-1\\3\end{bmatrix}$ is in the span, while $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is not.

## 6 Matrix multiplication

**Definition 39.** We say that $A$ is a $m\times n$ matrix if it has $m$ rows and $n$ columns.

**Example 40.** $\begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}$ is a $2\times 3$ matrix.

We have already seen three ways of writing systems of linear equations:

$$\left[\begin{array}{cc|c}2&3&1\\-1&1&-3\end{array}\right]\quad\begin{cases}\text{row picture} & \begin{array}{rcrcr}2x_1&+&3x_2&=&1\\-x_1&+&x_2&=&-3\end{array}\\[2ex]\text{column picture} & x_1\begin{bmatrix}2\\-1\end{bmatrix}+x_2\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}1\\-3\end{bmatrix}\end{cases}$$

We now wish to write linear systems simply as $A\boldsymbol{x}=\boldsymbol{b}$. Here, $\begin{bmatrix}2&3\\-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\-3\end{bmatrix}$.

**Example 41.** For this, we need $\begin{bmatrix}2&3\\-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}2\\-1\end{bmatrix}+x_2\begin{bmatrix}3\\1\end{bmatrix}$.

**Definition 42.** The product of a matrix $A$ with a vector $\boldsymbol{x}$ is a linear combination of the columns of $A$ with weights given by the entries of $\boldsymbol{x}$. In other words,

$$A\boldsymbol{x}=x_1\begin{pmatrix}\text{col } 1\\\text{of } A\end{pmatrix}+x_2\begin{pmatrix}\text{col } 2\\\text{of } A\end{pmatrix}+\cdots$$

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**Example 43.**

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

**Example 44.**

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 12 & -11 \end{bmatrix}$

because $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$

and $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 12 & -11 & 5 \end{bmatrix}$

**Definition 45.** Each column of $AB$ is a linear combination of the columns of $A$ with weights given by the corresponding column of $B$. For instance,

$$\begin{bmatrix} \text{col } 5 \\ \text{of} \\ AB \end{bmatrix} = A \cdot \begin{bmatrix} \text{col } 5 \\ \text{of} \\ B \end{bmatrix}.$$

**Example 46.**

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$

This is the $2 \times 2$ **identity matrix**.

**Theorem 47.** Let $A, B, C$ be matrices of appropriate size. Then:

- $A(BC) = (AB)C$ associative
- $A(B + C) = AB + AC$ left-distributive
- $(A + B)C = AC + BC$ right-distributive

**Example 48.** However, matrix multiplication is not commutative!

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}$

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