We never write $\frac{A}{B}$ for matrices! Why?

Because it is unclear whether you mean AB^{-1} or $B^{-1}A$. (And order matters a lot with matrices!)

7.1 Recipe for computing the inverse of any matrix

To compute A^{-1}: **Gauss–Jordan method**

- Form the augmented matrix $[\ A\,|\,I\ ]$.
- Compute the reduced echelon form. (i.e. Gauss–Jordan elimination)
- If A is invertible, the RREF is of the form $[\ I\,|\,A^{-1}\ ]$.

Why is that reasonable?

- Well, to solve $A\boldsymbol{x} = \boldsymbol{b}$, we do row reduction on $[\ A\,|\,\boldsymbol{b}\ ]$.
- Likewise, to solve $AX = I$ (to find the inverse X), we do row reduction on $[\ A\,|\,I\ ]$.

Example 59. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, if it exists.

Solution. We compute the RREF of $[\ A\,|\,I\ ]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_2 - 2R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Hence, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Comment. When computing any product BA, the matrix BA is obtained from B by the single column operation $C_1 + 2C_2 \Rightarrow C_1$ (why?!). That's the effect of multiplication with A (on the right). This can be "undone" by the single column operation $C_1 - 2C_2 \Rightarrow C_1$. The corresponding matrix is A^{-1}.

Example 60. Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution. We compute the RREF of $[\ A\,|\,I\ ]$:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_2 + \frac{3}{2}R_1 \Rightarrow R_2}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{\substack{\frac{1}{2}R_1 \Rightarrow R_1 \\ R_2 \Leftrightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

Hence, $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$. [We can easily check that: $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$]

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Example 61. Solve $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Then, solve $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

Solution. From the previous problem, we know that the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is invertible. Using its inverse, we find that (review Theorem 57)

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

is the unique solution.

Likewise, for the second system, we have the unique solution

$$\boldsymbol{x} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Important practical observation. Note how easy it is now (once we have the inverse) to solve linear systems for various right-hand sides.

7.2 The inverse of a 2×2 matrix

The following formula immediately gives us the inverse of a 2×2 matrix (if it exists). It is worth remembering!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{provided that } ad - bc \neq 0$$

Let's check that! $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -cb + ad \end{bmatrix} = I_2$

Note.

- A 1×1 matrix $[\ a\ ]$ is invertible $\iff a \neq 0$.
- A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad - bc \neq 0$.

We will later see that the quantities on the RHS are the **determinants** of these matrices.

Example 62. Solve $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by inverting the matrix.

Solution. We multiply both sides with the inverse of $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$.

[In the process of doing so, we see that the inverse does exist.]

Using the formula for 2×2 matrices:

$$\boldsymbol{x} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{14 - 15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$