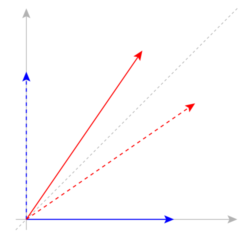
**Example 119.** Use your geometric understanding to find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



**Solution.** $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ (i.e. multiplication with $A$ is reflection through the line $y = x$)

- $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\rightsquigarrow$ $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 1$.
- $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\rightsquigarrow$ $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = -1$.

**Example 120.** Verify that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}$. What is its eigenvalue?

**Solution.** $A\boldsymbol{x} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} = 4\boldsymbol{x}$.
Hence, $\boldsymbol{x}$ is an eigenvector of $A$ with eigenvalue $4$.

## 15.1 How to solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$

**Example 121. (review)** If $B$ is invertible, what can you say about $\operatorname{null}(B)$?

**Solution.** If $B$ is invertible, then $B\boldsymbol{x} = \boldsymbol{0}$ has the unique solution $\boldsymbol{x} = \boldsymbol{0}$.
Hence, $\operatorname{null}(B) = \{\boldsymbol{0}\}$. (In particular, $\dim \operatorname{null}(B) = 0$.) In other words, the null space is trivial.

**Comment.** If you write down a random $n \times n$ matrix, then, most likely, it will be invertible (none of the pivots are $0$).

Key observation:

$$\begin{aligned} & A\boldsymbol{x} = \lambda\boldsymbol{x} \\ \iff & A\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0} \\ \iff & (A - \lambda I)\boldsymbol{x} = \boldsymbol{0} \end{aligned}$$

This homogeneous system has a nontrivial solution if and only if $\det(A - \lambda I) = 0$.

**Recipe.** To find eigenvectors and eigenvalues of $A$.

(a) First, find the eigenvalues $\lambda$ by solving $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is a polynomial in $\lambda$, called the **characteristic polynomial** of $A$.

(b) Then, for each eigenvalue $\lambda$, find corresponding eigenvectors by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$.

More precisely, we find a basis of eigenvectors for the **$\lambda$-eigenspace** $\operatorname{null}(A - \lambda I)$.

**Important comment.** If $A$ is $n \times n$, then $\det(A - \lambda I)$ is a polynomial in $\lambda$ of degree $n$. Such a polynomial has at most $n$ roots (it has exactly $n$ roots when working with complex numbers and counting multiplicity).

Hence, an $n \times n$ matrix $A$ has at most $n$ different eigenvalues!

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**Example 122.** Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

In other words: Find the eigenvalues of $A$ as well as bases for the corresponding eigenspaces.

**Solution.**

- $A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

  The **characteristic polynomial** of $A$ is:

  $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$

  Its roots are $\lambda_1 = 2$, $\lambda_2 = 4$, the **eigenvalues** of $A$.

- Find eigenvectors with eigenvalue $\lambda_1 = 2$: (we also call these $2$-eigenvectors)

  Basis for $\operatorname{null}(A - \lambda_1 I) = \operatorname{null}\left( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

  So: $\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = 2$.

[All other eigenvectors with $\lambda = 2$ are multiples of $\boldsymbol{x}_1$. $\operatorname{null}\left( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is the $2$-**eigenspace**.]

- Find eigenvectors with eigenvalue $\lambda_2 = 4$:

  Basis for $\operatorname{null}(A - \lambda_2 I) = \operatorname{null}\left( \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right)$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

  So: $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda_2 = 4$.

[All other eigenvectors with $\lambda = 4$ are multiples of $\boldsymbol{x}_2$. $\operatorname{null}\left( \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is the $4$-**eigenspace**.]

**Check it!** We check our answer:

- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \overset{\checkmark}{=} 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \overset{\checkmark}{=} 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**Comment.** In the first step, we needed to find the roots of $(3-\lambda)^2 - 1$. There are two nice options here:

- From Calculus you already know one method for finding the roots of any quadratic polynomial, which usually goes by $abc$-formula or $pq$-formula. Using the $abc$-formula, we find that the two roots of $(3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$ are

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36 - 32}}{2} = \frac{6 \pm 2}{2} \quad \rightsquigarrow \quad \lambda_1 = 2, \quad \lambda_2 = 4.$$

- In this particular example, it is easier to notice that $(3-\lambda)^2 - 1 = 0$, or $(3-\lambda)^2 = 1$ has the solutions

$$3 - \lambda = \pm 1 \quad \rightsquigarrow \quad \lambda_1 = 2, \quad \lambda_2 = 4.$$

  When working by hand, it is often worthwhile to watch out for such shortcuts before expanding the characteristic polynomial.