Preparing for Midterm #1

MATH 237 — Linear Algebra I
Thursday, Sept 29

Please print your name:

I Computational part

Problem 1. Compute the following, or state why it is not possible to do so:

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^T$

(b) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1}$

(c) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$

(d) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$

(e) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Solution.

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^T$

(b) Non-square matrices are not invertible.

(c) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{4\cdot 1 - 2\cdot 1}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}$

(d) This matrix is not invertible.

(e) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4 \\ 4 & 0 & 2 \\ 13 & -1 & 8 \end{bmatrix}$

(f) A 2×3 matrix and a 2×2 matrix cannot be multiplied (in that order). □

Problem 2. Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ -4 \\ -1 \\ 1 \end{bmatrix}.$$

(a) Are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ linearly independent? If not, write down a linear dependence relation.

(b) Is $\boldsymbol{v}_4$ in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$? If so, write $\boldsymbol{v}_4$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

(c) Are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ linearly independent? If not, write down a linear dependence relation.

(d) Is $\boldsymbol{v}_3$ in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$? If so, write $\boldsymbol{v}_3$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$.

Solution.

(a) We eliminate!

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{R_3-R_1\Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{\substack{\frac{1}{2}R_2\Rightarrow R_2 \\ R_3-\frac{1}{2}R_2\Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{R_4+R_3\Rightarrow R_4}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this echelon form, we see that there is a free variable. Hence, the system $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 + x_4\boldsymbol{v}_4 = \boldsymbol{0}$ has solutions besides the trivial one ($\boldsymbol{x} = \boldsymbol{0}$). This means that the four vectors are linearly dependent.

To write down the general solution we can either continue eliminating until we have the RREF, or we can solve the system by back-substitution, which is what we do here. We find $x_4 = s$, $x_3 = -s$, $x_2 = 2s$, $x_1 = -2x_4 - x_3 = -s$. In other words, the general solution is

$$\boldsymbol{x} = \begin{bmatrix} -s \\ 2s \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

This means that we have the linear dependence relation

$$-\boldsymbol{v}_1 + 2\boldsymbol{v}_2 - \boldsymbol{v}_3 + \boldsymbol{v}_4 = \boldsymbol{0}.$$

(This is the case $s = 1$. The most general linear dependence relation is $-s\boldsymbol{v}_1 + 2s\boldsymbol{v}_2 - s\boldsymbol{v}_3 + s\boldsymbol{v}_4 = \boldsymbol{0}$ but that contains no extra information, because it is just the previous relation multiplied with s.)

(b) Yes, $\boldsymbol{v}_4$ is in span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Our previous computation shows that $\boldsymbol{v}_4 = \boldsymbol{v}_1 - 2\boldsymbol{v}_2 + \boldsymbol{v}_3$.

(c) Our computation in the first part (just forget about the fourth column) shows that the system $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{x} = \boldsymbol{0}$ has no free variables. Hence, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent.

(d) No. Because $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent, it is not possible to write $\boldsymbol{v}_3 = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2$ (because then we would have the dependence relation $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 - \boldsymbol{v}_3 = \boldsymbol{0}$). □

Problem 3. Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}.$$

(a) For which value(s) of h is $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

(b) For which value(s) of h are $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ linearly independent?

Solution.

(a) We have to determine whether the following system has a solution:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 3 & h \end{array}\right] \overset{\substack{-R_2\Rightarrow R_2 \\ R_3+R_1\Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & h+1 \end{array}\right] \overset{R_3-3R_2\Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & h+1 \end{array}\right]$$

This system is consistent if and only if $h = -1$. Hence, $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ for $h = -1$ only.

(b) Since $\boldsymbol{v}_1, \boldsymbol{v}_2$ are clearly independent (just two vectors that are not multiples of each other), the three vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ can be dependent only if $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (can you see that?). By the first part, this happens for $h = -1$ only. Hence, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent for all $h \neq -1$.

Alternative. If you didn't notice that, you would have to determine whether the system $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 3 & h \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$ has solutions apart from the trivial one. As above (we actually the same matrix because we are not spelling out the augmented part, since that part is always just all zeros),

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 3 & h \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & h+1 \end{bmatrix}$$

and we see that there is 3 pivots (and no free variables) unless $h = -1$. Again, we find that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent for all $h \neq -1$. □

Problem 4. Consider $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$.

(a) Determine A^{-1}.

(b) Using (a), solve $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

(c) Using your work in (a), determine $\det(A)$.

Solution.

(a)
$$\left[\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \Leftrightarrow R_2}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_3 - 2R_1 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 0 & -2 & 1 \end{array}\right]$$

$$\overset{R_3 + R_2 \Rightarrow R_3}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & -2 & 1 \end{array}\right] \overset{\substack{\frac{1}{3}R_2 \Rightarrow R_2 \\ \frac{1}{4}R_3 \Rightarrow R_3}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right] \overset{\substack{R_1 + R_3 \Rightarrow R_1 \\ R_2 - \frac{1}{3}R_3 \Rightarrow R_2}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{6} & -\frac{1}{12} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right]$$

$$\overset{R_1 - R_2 \Rightarrow R_1}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{6} & -\frac{1}{12} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right]$$

Hence, $\begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} & -\frac{1}{12} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

(b) $\boldsymbol{x} = A^{-1} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} & -\frac{1}{12} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{11}{12} \\ -\frac{7}{4} \end{bmatrix}$

(c) $\det\left(\begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\right) = -\det\left(\begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right) = -1 \cdot 3 \cdot 4 = -12$

Note that there was one exchange of rows when we eliminated from A to the echelon form. That's why we have a minus sign. All other elementary row operations were of type "add", which does not change the determinant.

□

Problem 5. Consider $B = \begin{bmatrix} 1 & 2 & 6 & 5 & -5 & 0 \\ 2 & 4 & 14 & 12 & -12 & -2 \\ 1 & 2 & 4 & 3 & -2 & 6 \end{bmatrix}$.

(a) Determine the row-reduced echelon form of B.

(b) Use your result in (a) to find the general solution of the linear system:

$$\begin{array}{rrrrrcr} x_1 & +2x_2 & +6x_3 & +5x_4 & -5x_5 & = & 0 \\ 2x_1 & +4x_2 & +14x_3 & +12x_4 & -12x_5 & = & -2 \\ x_1 & +2x_2 & +4x_3 & +3x_4 & -2x_5 & = & 6 \end{array}$$

(c) Determine the general solution to the associated homogeneous linear system, that is:

$$\begin{array}{rrrrrcr} x_1 & +2x_2 & +6x_3 & +5x_4 & -5x_5 & = & 0 \\ 2x_1 & +4x_2 & +14x_3 & +12x_4 & -12x_5 & = & 0 \\ x_1 & +2x_2 & +4x_3 & +3x_4 & -2x_5 & = & 0 \end{array}$$

(d) Are the columns of $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 \\ 2 & 4 & 14 & 12 & -12 \\ 1 & 2 & 4 & 3 & -2 \end{bmatrix}$ linearly independent?

If not, write down a non-trivial linear combination of the columns, which produces $\mathbf{0}$.

Solution.

(a) $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 & 0 \\ 2 & 4 & 14 & 12 & -12 & -2 \\ 1 & 2 & 4 & 3 & -2 & 6 \end{bmatrix} \overset{\substack{R_2-2R_1 \Rightarrow R_2 \\ R_3-R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 6 & 5 & -5 & 0 \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & -2 & -2 & 3 & 6 \end{bmatrix} \overset{R_3+R_2 \Rightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 6 & 5 & -5 & 0 \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

$\overset{R_1-3R_2 \Rightarrow R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & -1 & 1 & 6 \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \overset{\substack{R_1-R_3 \Rightarrow R_1 \\ R_2+2R_3 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \overset{\frac{1}{2}R_2 \Rightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

(b) x_2 and x_4 are free variables, and so we set $x_2 = s_1$ and $x_4 = s_2$. Then, the general solution is

$$\begin{aligned} x_1 &= 2 - 2s_1 + s_2 \\ x_2 &= s_1 \\ x_3 &= 3 - s_2 \\ x_4 &= s_2 \\ x_5 &= 4 \end{aligned}$$

where s_1, s_2 can take any value. In vector form, the solution is

$$\boldsymbol{x} = \begin{bmatrix} 2-2s_1+s_2 \\ s_1 \\ 3-s_2 \\ s_2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 4 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

(Unless specifically asked, it is up to us how we prefer to write the solution.)

(c) Note that we don't need to redo any calculations! The right-hand side is replaced with all zeros, which throught the elimination will just remain all zeros.

As such, the RREF of the augmented matrix will change from $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$ to $\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$.

The general solution is

$$\begin{aligned} x_1 &= -2s_1+s_2 \\ x_2 &= s_1 \\ x_3 &= -s_2 \\ x_4 &= s_2 \\ x_5 &= 0 \end{aligned}$$

where s_1, s_2 can take any value.

(d) They are definitely linearly dependent, because these are 5 vectors in $\mathbb{R}^3$.

To write down a linear dependence relation, we have to solve the system $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 \\ 2 & 4 & 14 & 12 & -12 \\ 1 & 2 & 4 & 3 & -2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. But that is just the homogeneous system that we have already solved in part (c) and used in the other parts.

We therefore have the following very general linear dependence relation for the columns of our matrix:

$$(-2s_1+s_2)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} - s_2 \begin{bmatrix} 6 \\ 14 \\ 4 \end{bmatrix} + s_2 \begin{bmatrix} 5 \\ 12 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} -5 \\ -12 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Comment. If we want to just write down a specific dependence relation, we can choose s_1 and s_2 be any numbers we like. There are two canonical choices, which lead to very different dependence relations: one choice is $s_1 = 1$ and $s_2 = 0$, and the other choice is $s_1 = 0$ and $s_2 = 1$. The corresponding linear dependence relations are

$$-2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 6 \\ 14 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 12 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} -5 \\ -12 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 6 \\ 14 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 12 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} -5 \\ -12 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

□

Problem 6. Evaluate the following determinants. [Real computations only necessary for the last two.]

(a) $\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$

(f) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix}$

Solution.

(a) $\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix} = 0$ because the columns are not linearly independent. (Column one and two are the same.)

(b) $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 2 \cdot 6 = 12$

(c) $\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = (-1)(-1)(-1) = -1$ because it takes three row interchanges ($R_1 \Leftrightarrow R_6$, $R_2 \Leftrightarrow R_5$, $R_3 \Leftrightarrow R_4$) to transform this matrix to the 6×6 identity matrix.

(d) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{vmatrix} = 0$ because the matrix is clearly not invertible. (Look at the last column!)

(e) $$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} \overset{\substack{R_2 - R_1 \Rightarrow R_2 \\ R_3 - 3R_1 \Rightarrow R_3}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & -4 & -8 \end{vmatrix} \overset{R_3 - 4R_2 \Rightarrow R_3}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -8 \end{vmatrix} = 1 \cdot (-1) \cdot (-8) = 8$$

(f) $$\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} \overset{\substack{R_2 - 2R_1 \Rightarrow R_2 \\ R_3 + R_1 \Rightarrow R_3}}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix} \overset{R_4 + 2R_2 \Rightarrow R_4}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{vmatrix} \overset{R_4 + \frac{5}{2}R_3 \Rightarrow R_4}{=} \begin{vmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$
$= 1 \cdot (-1) \cdot (-2) \cdot 10 = 20$ □

II Short answer part

Problem 7. Let A be a $p \times q$ matrix and B be an $r \times s$ matrix. Under which condition is $A^T B$ defined?

Solution. A^T is $q \times p$ and B is $r \times s$. Hence, we need $p = r$. □

Problem 8. Decide whether the following statements are true or false.

(a) If A is invertible then the system $A\boldsymbol{x} = \boldsymbol{b}$ always has the same number of solutions.

(b) The homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ is always consistent.

(c) In order for A to be invertible, the matrix A has to be square (that is, of shape $n \times n$).

(d) If A is a 4×3 matrix with 2 pivot columns, then the columns of A are linearly independent.

(e) If A is invertible then the columns of A are linearly independent.

(f) $\boldsymbol{b}$ is in the span of the columns of A if and only if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

(g) Every matrix can be reduced to echelon form by a sequence of elementary row operations.

(h) The row-reduced echelon form of a matrix is unique.

Solution.

(a) True. $A\boldsymbol{x} = \boldsymbol{b}$ always has exactly one solution.

(b) True. It has the trivial solution $\boldsymbol{x} = \boldsymbol{0}$.

(c) True.

(d) False. Since A has 3 columns but only 2 pivots, the system $A\boldsymbol{x} = \boldsymbol{0}$ has a free variable. This means that $A\boldsymbol{x} = \boldsymbol{0}$ has solutions beside the trivial one, and so the columns of A are linearly dependent.

(e) True, because $A\boldsymbol{x} = \boldsymbol{0}$ only has the unique (trivial) solution $x = A^{-1}\boldsymbol{0} = \boldsymbol{0}$.

(f) True. Keep in mind that $A\boldsymbol{x}$ is a linear combination of the columns of A.

(g) True. And we can do it! All the way to a RREF if necessary.

(h) True. □

Problem 9. We are solving a linear system with 4 equations and 5 unknowns. Which of the following are possible?

(a) The system has no solution.

(b) The system has a unique solution.

(c) The system has infinitely many solutions.

Solution. The corresponding matrix A has 4 rows and 5 columns. Since there is only room for 4 pivots, there has to be at least one free variable. Hence, the system cannot have a unique solution. The other two options are possible. □

Problem 10. We are solving a linear system with 5 equations and 5 unknowns. Which of the following are possible?

(a) The system has no solution.

(b) The system has a unique solution.

(c) The system has infinitely many solutions.

Solution. All of these are possible. □

Problem 11. For which values of a is the matrix $\begin{bmatrix} 3 & a-6 \\ 3a & -a+6 \end{bmatrix}$ invertible?

Solution. Recall that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is invertible if and only if $ad - bc \neq 0$. Hence, the matrix above is invertible if and only if

$$3(-a+6) - (a-6)(3a) = 3(a^2 + 5a + 6) = 3(a+2)(a+3) \neq 0.$$

We conclude that the matrix is invertible for all values of a except $a = -2$ and $a = -3$. □

Problem 12. What is the augmented matrix for the following system of linear equations?

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_2 - x_3 &= 2 \\ x_1 + x_2 &= 3 \end{aligned}$$

Solution. $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \end{array}\right]$ □

Problem 13. List the three kinds of elementary row operations. Give an example for each kind using the shorthand notation that we use in class.

Solution.

- *(add)* Add a multiple of one row to another. For instance, $R_2 + 3R_1 \Rightarrow R_2$.
- *(scale)* Multiply a row by a nonzero constant. For instance, $\frac{1}{2}R_1 \Rightarrow R_1$.
- *(swap)* Interchange two rows. For instance, $R_1 \Leftrightarrow R_2$.

□

Problem 14. Decide whether the following vectors are linearly independent. No computations necessary!

(a) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ☐ dependent ☐ independent

(b) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ ☐ dependent ☐ independent

(c) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ ☐ dependent ☐ independent

Solution.

(a) dependent (because of the zero vector)

(b) independent (because the two vectors are not multiples of each other)

(c) dependent (because these are four vectors in $\mathbb{R}^3$) □

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Problem 15. Write down the cofactor expansion for the determinant of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ along

(a) the second row,

(b) the third column.

Solution.

(a) $\det(A) = -d \cdot \begin{bmatrix} b & c \\ h & i \end{bmatrix} + e \cdot \begin{bmatrix} a & c \\ g & i \end{bmatrix} - f \cdot \begin{bmatrix} a & b \\ g & h \end{bmatrix}$

(b) $\det(A) = c \cdot \begin{bmatrix} d & e \\ g & h \end{bmatrix} - f \cdot \begin{bmatrix} a & b \\ g & h \end{bmatrix} + i \cdot \begin{bmatrix} a & b \\ d & e \end{bmatrix}$ □

Problem 16. If A and B are 3×3 matrices with $\det(A) = 4$ and $\det(B) = -1$. What is the determinant of $C = 2A^T A^{-1} B A$?

Solution. We have

$$\det(C) = 2^3 \det(A^T) \det(A^{-1}) \det(B) \det(A) = 8 \det(A) \frac{1}{\det(A)} \det(B) \det(A) = 8 \det(A) \det(B) = -32.$$

□

Problem 17. Let A be a $n \times n$ matrix with $A^T = A^{-1}$. What can you say about $\det(A)$?

Solution. If $A^T = A^{-1}$, then $\det(A^T) = \det(A^{-1})$ or, simplified, $\det(A) = \frac{1}{\det(A)}$. It follows that $(\det(A))^2 = 1$, which implies that $\det(A) = 1$ or $\det(A) = -1$. □

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