Preparing for Midterm #1

MATH 237 — Linear Algebra I
Thursday, Sept 29

Please print your name:

I Computational part

Problem 1. Compute the following, or state why it is not possible to do so:

(a) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^T$

(b) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}^{-1}$

(c) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$

(d) $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$

(e) $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Problem 2. Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ -4 \\ -1 \\ 1 \end{bmatrix}.$$

(a) Are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ linearly independent? If not, write down a linear dependence relation.

(b) Is $\boldsymbol{v}_4$ in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$? If so, write $\boldsymbol{v}_4$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

(c) Are the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ linearly independent? If not, write down a linear dependence relation.

(d) Is $\boldsymbol{v}_3$ in $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$? If so, write $\boldsymbol{v}_3$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$.

Problem 3. Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}.$$

(a) For which value(s) of h is $\boldsymbol{v}_3$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?

(b) For which value(s) of h are $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ linearly independent?

Problem 4. Consider $A = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$.

(a) Determine A^{-1}.

(b) Using (a), solve $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

(c) Using your work in (a), determine $\det(A)$.

Armin Straub
straub@southalabama.edu

Problem 5. Consider $B = \begin{bmatrix} 1 & 2 & 6 & 5 & -5 & 0 \\ 2 & 4 & 14 & 12 & -12 & -2 \\ 1 & 2 & 4 & 3 & -2 & 6 \end{bmatrix}$.

(a) Determine the row-reduced echelon form of B.

(b) Use your result in (a) to find the general solution of the linear system:

$$\begin{array}{rrrrrcr} x_1 & +2x_2 & +6x_3 & +5x_4 & -5x_5 & = & 0 \\ 2x_1 & +4x_2 & +14x_3 & +12x_4 & -12x_5 & = & -2 \\ x_1 & +2x_2 & +4x_3 & +3x_4 & -2x_5 & = & 6 \end{array}$$

(c) Determine the general solution to the associated homogeneous linear system, that is:

$$\begin{array}{rrrrrcr} x_1 & +2x_2 & +6x_3 & +5x_4 & -5x_5 & = & 0 \\ 2x_1 & +4x_2 & +14x_3 & +12x_4 & -12x_5 & = & 0 \\ x_1 & +2x_2 & +4x_3 & +3x_4 & -2x_5 & = & 0 \end{array}$$

(d) Are the columns of $\begin{bmatrix} 1 & 2 & 6 & 5 & -5 \\ 2 & 4 & 14 & 12 & -12 \\ 1 & 2 & 4 & 3 & -2 \end{bmatrix}$ linearly independent?

If not, write down a non-trivial linear combination of the columns, which produces $\mathbf{0}$.

Problem 6. Evaluate the following determinants. [Real computations only necessary for the last two.]

(a) $\begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 6 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 2 & 5 & 0 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$

(f) $\begin{vmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{vmatrix}$

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II Short answer part

Problem 7. Let A be a $p \times q$ matrix and B be an $r \times s$ matrix. Under which condition is $A^T B$ defined?

Problem 8. Decide whether the following statements are true or false.

(a) If A is invertible then the system $A\boldsymbol{x} = \boldsymbol{b}$ always has the same number of solutions.

(b) The homogeneous system $A\boldsymbol{x} = \mathbf{0}$ is always consistent.

(c) In order for A to be invertible, the matrix A has to be square (that is, of shape $n \times n$).

(d) If A is a 4×3 matrix with 2 pivot columns, then the columns of A are linearly independent.

(e) If A is invertible then the columns of A are linearly independent.

(f) $\boldsymbol{b}$ is in the span of the columns of A if and only if the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

(g) Every matrix can be reduced to echelon form by a sequence of elementary row operations.

(h) The row-reduced echelon form of a matrix is unique.

Problem 9. We are solving a linear system with 4 equations and 5 unknowns. Which of the following are possible?

(a) The system has no solution.

(b) The system has a unique solution.

(c) The system has infinitely many solutions.

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(a) The system has no solution.

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Problem 11. For which values of a is the matrix $\begin{bmatrix} 3 & a-6 \\ 3a & -a+6 \end{bmatrix}$ invertible?

Problem 12. What is the augmented matrix for the following system of linear equations?

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_2 - x_3 &= 2 \\ x_1 + x_2 &= 3 \end{aligned}$$

Problem 13. List the three kinds of elementary row operations. Give an example for each kind using the shorthand notation that we use in class.

Problem 14. Decide whether the following vectors are linearly independent. No computations necessary!

(a) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ☐ dependent ☐ independent

(b) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ ☐ dependent ☐ independent

(c) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ ☐ dependent ☐ independent

Problem 15. Write down the cofactor expansion for the determinant of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ along

(a) the second row,

(b) the third column.

Problem 16. If A and B are 3×3 matrices with $\det(A) = 4$ and $\det(B) = -1$. What is the determinant of $C = 2A^T A^{-1} B A$?

Problem 17. Let A be a $n \times n$ matrix with $A^T = A^{-1}$. What can you say about $\det(A)$?

Armin Straub
straub@southalabama.edu