Preparing for Midterm #2

MATH 237 — Linear Algebra I
Thursday, Nov 10

Please print your name:

I Computational part

Problem 1. In each case, find a basis for $\mathrm{col}(A), \mathrm{row}(A), \mathrm{null}(A)$.

(a) $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -1 & -2 & -1 & -1 & -3 \\ 2 & 4 & 0 & -6 & 7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Solution.

(a) Our first step is to bring A into RREF (just an echelon form would be enough, but then we would need to back-substitute when solving $A\boldsymbol{x} = \mathbf{0}$ for $\mathrm{null}(A)$):

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 5 \\ -1 & -2 & -1 & -1 & -3 \\ 2 & 4 & 0 & -6 & 7 \end{bmatrix} \underset{\text{do it!}}{\overset{\text{RREF}}{\rightsquigarrow}} \begin{bmatrix} 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- A basis for $\mathrm{col}(A)$ is: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}$. $(\dim \mathrm{col}(A) = 3)$
- A basis for $\mathrm{row}(A)$ is: $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. $(\dim \mathrm{row}(A) = 3)$
- $x_2 = s_1$ and $x_4 = s_2$ are our free variables. The general solution to $A\boldsymbol{x} = \mathbf{0}$ is:

$$\boldsymbol{x} = \begin{bmatrix} -2s_1 + 3s_2 \\ s_1 \\ -4s_2 \\ s_2 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

Hence, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{null}(A)$. $(\dim \mathrm{null}(A) = 2)$

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(b) A basis for col(A) is: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

A basis for row(A) is: $[\ 1\ ]$.

null(A) $= \{[\ 0\ ]\}$ (only the trivial solution), which has dimension 0 and therefore a basis with 0 vectors (that is, a/the basis is the empty set $\emptyset$).

(c) A basis for col(A) is: $[\ 1\ ]$.

A basis for row(A) is: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

The general solution to $A\boldsymbol{x} = \boldsymbol{0}$ is (note that A is in RREF already) $\boldsymbol{x} = \begin{bmatrix} -2s_1 - 3s_2 \\ s_1 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

Hence, a basis for null(A) is: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. □

Problem 2. Find the eigenvalues and bases for the eigenspaces of the following matrices.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 1 & 6 \\ 2 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution.

(a) By expanding by the first row, twice, we find that the characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 4-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 \begin{vmatrix} 4-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3(4-\lambda).$$

The eigenvalues are $\lambda = 1$ (with multiplicity 3) and $\lambda = 4$.

- For $\lambda = 4$, the eigenspace is null$\left(\begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

  [If you don't just see that, do the row operations to find the basis of the null space!]

- For $\lambda = 1$, the eigenspace is null$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$.

  [Here we can just "see" this basis. Just to make sure, also do the row operations to find the basis!]

(b) By expanding by the second column, we find that the characteristic polynomial is

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 1-\lambda & 6 \\ 2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & -2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(-\lambda(4-\lambda)+4) = (1-\lambda)(\lambda-2)^2.$$

The eigenvalues are $\lambda=1$ and $\lambda=2$ (with multiplicity 2).

- For $\lambda=1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -1 & 0 & -2 \\ 1 & 0 & 6 \\ 2 & 0 & 3 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

- For $\lambda=2$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} -2 & 0 & -2 \\ 1 & -1 & 6 \\ 2 & 0 & 2 \end{bmatrix}\right)$.

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & -1 & 6 \\ 2 & 0 & 2 \end{bmatrix} \overset{\substack{R_2+\frac{1}{2}R_1 \Rightarrow R_2 \\ R_3+R_1 \Rightarrow R_3}}{\rightsquigarrow} \begin{bmatrix} -2 & 0 & -2 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \overset{\substack{-\frac{1}{2}R_1 \Rightarrow R_1 \\ -R_2 \Rightarrow R_2}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

We see from here that $\operatorname{null}\left(\begin{bmatrix} -2 & 0 & -2 \\ 1 & -1 & 6 \\ 2 & 0 & 2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$.

[Note that it wasn't clear initially whether the eigenspace would have dimension 1 or dimension 2.]

(c) By expanding by the first column, we find that the characteristic polynomial is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = -\lambda^3.$$

The only eigenvalue is $\lambda=0$ (with multiplicity 3).

- For $\lambda=0$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(d) By expanding by the first column, we find that the characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = (1-\lambda)\lambda^2.$$

The eigenvalues are $\lambda=0$ (with multiplicity 2) and $\lambda=1$.

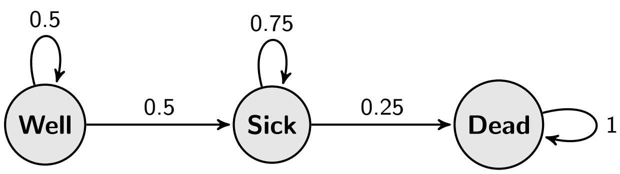
- For $\lambda=1$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- For $\lambda=0$, the eigenspace is $\operatorname{null}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, which has basis $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

□

Problem 3. Suppose there is an epidemic in which, every month, half of those who are well become sick, and a quarter of those who are sick become dead. (In this sad example, there is no recoveries, no resurrections, and no deaths without getting sick first.) What is the proportion of dead people in the long term equilibrium?

Solution. The transition graph is:



x_t: proportion of those well at time t (in months)

y_t: proportion of those sick at time t

z_t: proportion of those dead at time t

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0.5x_t \\ 0.5x_t + 0.75y_t \\ 0.25y_t + z_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.75 & 0 \\ 0 & 0.25 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

$\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix}$ is an equilibrium if $\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0.75 & 0 \\ 0 & 0.25 & 1 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix}$. That is, $\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix}$ is an eigenvector with eigenvalue 1.

The 1-eigenspace $\operatorname{null}\left(\begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (Do row operations if you don't see that right away!)

Since $x_\infty + y_\infty + z_\infty = 1$ in $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we conclude that $\begin{bmatrix} x_\infty \\ y_\infty \\ z_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Hence, the proportion of dead people is $1 = 100\%$ in the long term equilibrium.

Comment. Think about why this answer makes sense and is exactly what you expect without knowing about Markov chains and our kinds of computations.

Important comment. Also review the example we did in class. □

Problem 4. Consider $H = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

(a) Give a basis for H. What is the dimension of H?

(b) Determine whether the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in H. What about the vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$? If possible, express each vector in terms of your basis for H.

(c) Extend your basis of H to a basis of $\mathbb{R}^3$.

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Solution.

(a) Since $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 3 \\ 2 & 4 & 4 \end{bmatrix} \underset{\rightsquigarrow}{\overset{R_2-R_1\Rightarrow R_2}{R_3-2R_1\Rightarrow R_3}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we conclude that $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ is a basis for H.

In particular, $\dim H = 2$.

(b) We need to solve $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Let us do both at the same time (by working with two right-hand sides at once):

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 2 & 4 & 0 & 2 \end{array}\right] \underset{\rightsquigarrow}{\overset{R_2-R_1\Rightarrow R_2}{R_3-2R_1\Rightarrow R_3}} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right] \underset{\rightsquigarrow}{-1R_2\Rightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 0 \end{array}\right] \underset{\rightsquigarrow}{R_1-2R_2\Rightarrow R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

The first system is inconsistent and so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in H. On the other hand, the second system is consistent and so $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is in H. This much was visible right after the first step of elimination.

From the RREF, we read off that $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

(c) We need to add a third vector to our basis $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ of H. In the previous part, we found that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in H.

In other words, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent.

It follows that $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are a basis for $\mathbb{R}^3$.

□

Problem 5. Is it true that $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$?

Solution. Let $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$.

We check that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \in V$ and $\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \in V$.

This follows from $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ -1 & -2 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \underset{\rightsquigarrow}{\overset{R_2+R_1\Rightarrow R_2}{R_3-R_1\Rightarrow R_3}} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \underset{\rightsquigarrow}{R_4+\frac{1}{2}R_2\Rightarrow R_4} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, because for both right-hand sides the system is consistent.

Since these two vectors span W, this implies that W is a subspace of V.

But both spaces have dimension 2, and so they must be equal: $V = W$. □

II Short answer part

Problem 6. In each case, write down a precise definition or answer.

(a) What is a vector space?

(b) What is the rank of a matrix?

(c) What does it mean for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ from a vector space to be linearly independent?

(d) What does it mean for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ to be a basis for a vector space V?

Solution.

(a) A vector space is a set V of vectors that can be written as a span (that is, $V = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots\}$ for a bunch of vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$).

An alternative, more abstract, definition is: A vector space is a set V of vectors such that

- if $\boldsymbol{v}, \boldsymbol{w} \in V$, then $\boldsymbol{v} + \boldsymbol{w} \in V$, [closed under addition]
- if $\boldsymbol{v} \in V$ and $r \in \mathbb{R}$, then $r\boldsymbol{v} \in V$. [closed under scalar multiplication]

Moreover, the addition and scalar multiplication needs to satisfy the laws we expect from them.

(b) The rank of a matrix is the number of pivots in an echelon form.

Alternatively: The rank of a matrix is the dimension of its column space. (Or, row space.)

(c) Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent if the only solution to

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_n\boldsymbol{v}_n = \boldsymbol{0}$$

is the trivial one ($x_1 = x_2 = \ldots = x_n = 0$).

(d) The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are a basis for V, if $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are linearly independent and $V = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

□

Problem 7. Decide whether the following sets of vectors are a basis of $\mathbb{R}^3$.

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

(b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ☐ basis of $\mathbb{R}^3$ ☐ not a basis of $\mathbb{R}^3$

Solution.

(a) basis

(b) not a basis

(c) not a basis

(d) not a basis □

Problem 8. True or false?

(a) Every vector space has a basis.

(b) The zero vector can never be a basis vector.

(c) Every set of linearly independent vectors in V can be extended to a basis of V.

(d) $\mathrm{col}(A)$ and $\mathrm{row}(A)$ always have the same dimension.

(e) If B is the RREF of A, then we always have $\mathrm{col}(A) = \mathrm{col}(B)$.

(f) If B is the RREF of A, then we always have $\mathrm{row}(A) = \mathrm{row}(B)$.

(g) If B is the RREF of A, then we always have $\mathrm{null}(A) = \mathrm{null}(B)$.

(h) If a subspace V of $\mathbb{R}^3$ contains three linearly independent vectors, then always $V = \mathbb{R}^3$.

(i) There are matrices A such that $\mathrm{null}(A)$ is the empty set.

Solution.

(a) True. In fact, for all the spaces we can get our hands on, we know how to compute a basis.

[In the case of very infinite-dimensional spaces, this becomes "the axiom of choice".]

(b) True. A set of vectors that includes the zero vector can never be linearly independent.

(c) True. We just keep adding missing vectors from V to the initial set of linearly independent vectors until we span all of V. (If V has dimension d, then this process of adding vectors has to stop once we have a total of d vectors.)

(d) True.

(e) False. Elementary row operations do not preserve column spaces (except by accident).

For instance, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \overset{R_1 \Leftrightarrow R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but $\mathrm{col}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \neq \mathrm{col}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$.

(f) True. Elementary row operations do preserve the row space.

(g) True. Elementary row operations preserve the null space. (That's the reason, in abstract words, why we can do row operations to solve systems $A\boldsymbol{x} = \mathbf{0}$.)

(h) True. Three linearly independent vectors in $\mathbb{R}^3$ automatically form a basis of $\mathbb{R}^3$.

(i) False. $\mathrm{null}(A)$ always contains at least the zero vector (the trivial solution to $A\boldsymbol{x} = \mathbf{0}$). □

Problem 9.

(a) What is $\dim\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right)$?

(b) If $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}\right\}$, then $W = \operatorname{row}(A)$ with $A = \ldots$

(c) $\boldsymbol{v}$ is in $\operatorname{null}(A)$ if and only if ...

(d) Let A be a 5×5 matrix with $\dim\operatorname{row}(A) = 5$. What can you say about $\det(A)$?

(e) Let A be a 7×7 matrix with $\dim\operatorname{null}(A) = 1$. What can you say about $\det(A)$?

(f) What are the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 4 \end{bmatrix}$?

(g) Suppose V and W are subspaces of $\mathbb{R}^n$, and that $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a basis for V, and $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ is a basis for W. What can you say about $\dim U$ with $U = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$?

(h) Let A be a 4×3 matrix, whose row space has dimension 2. What is the dimension of $\operatorname{null}(A)$?

(i) Let A be a 3×3 matrix, whose column space has dimension 3. If $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$, what can you say about the number of solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$?

(j) Let A be a 3×3 matrix, whose column space has dimension 2. What can you say about $\det(A)$?

Solution.

(a) The rank of that matrix is clearly 1. Hence, $\dim\operatorname{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right) = 3 - 1 = 2$.

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 3 \end{bmatrix}$

(c) $\boldsymbol{v}$ is in $\operatorname{null}(A)$ if and only if $A\boldsymbol{v} = \boldsymbol{0}$.

(d) Since $\dim\operatorname{row}(A) = 5$ and A is 5×5, the matrix A is invertible. Therefore, $\det(A) \neq 0$.

(e) The null space of A contains a nonzero vector. Hence, $A\boldsymbol{x} = \boldsymbol{0}$ has a nonzero solution, that is, A is not invertible. Therefore, $\det(A) = 0$.

(f) The characteristic polynomial is $\det\begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ -1 & 3-\lambda & 0 & 0 \\ -1 & 1 & 3-\lambda & 0 \\ 0 & 1 & 2 & 4-\lambda \end{bmatrix} = (2-\lambda)(3-\lambda)^2(4-\lambda)$.

Hence, the eigenvalues are 2, 3 (with multiplicity 2) and 4.

[We see from here that, for any triangular matrix, the eigenvalues are just its diagonal entries.]

(g) $\dim U \in \{3, 4, 5\}$

(h) $\dim\operatorname{null}(A) = 3 - 2 = 1$

(i) If A is a 3×3 matrix, whose column space has dimension 3, then A is invertible.

Therefore, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any $\boldsymbol{b}$.

(j) If A is a 3×3 matrix, whose column space has dimension 2, then A is not invertible.

Therefore, $\det(A) = 0$. □

Problem 10. Suppose that the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Find a basis for each of $\operatorname{col}(A)$, $\operatorname{row}(A)$ and $\operatorname{null}(A)$.

Solution.

- A basis for $\operatorname{col}(A)$ is $\begin{bmatrix} a \\ d \\ g \end{bmatrix}, \begin{bmatrix} c \\ f \\ i \end{bmatrix}$.
- A basis for $\operatorname{row}(A)$ is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- A basis for $\operatorname{null}(A)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. □

Problem 11. Let A, B be $n \times n$ matrices such that $AB = \mathbf{0}$. Show that $\det(A) = 0$ or $\det(B) = 0$.

[Recall that it does not follow that $A = \mathbf{0}$ or $B = \mathbf{0}$.]

Solution. Obviously, $\det(AB) = 0$. On the other hand, $\det(AB) = \det(A)\det(B)$. It follows from $\det(A)\det(B) = 0$ that $\det(A) = 0$ or $\det(B) = 0$. □

Problem 12. If A has eigenvector $\boldsymbol{v}$ with eigenvalue λ, what can you say about eigenvalues and eigenvectors of:

(a) $7A$

(b) A^3

(c) $A - 2I$

Solution.

(a) Since $A\boldsymbol{v} = \lambda\boldsymbol{v}$, we have $7A\boldsymbol{v} = 7\lambda\boldsymbol{v}$. In other words, $\boldsymbol{v}$ is a 7λ-eigenvector of $7A$.

(b) Since $A\boldsymbol{v} = \lambda\boldsymbol{v}$, we have $A^3\boldsymbol{v} = AAA\boldsymbol{v} = AA\lambda\boldsymbol{v} = A\lambda^2\boldsymbol{v} = \lambda^3\boldsymbol{v}$. In other words, $\boldsymbol{v}$ is a λ^3-eigenvector of A^3.

(c) Since $A\boldsymbol{v} = \lambda\boldsymbol{v}$, we have $(A - 2I)\boldsymbol{v} = A\boldsymbol{v} - 2I\boldsymbol{v} = \lambda\boldsymbol{v} - 2\boldsymbol{v} = (\lambda - 2)\boldsymbol{v}$. In other words, $\boldsymbol{v}$ is a $\lambda - 2$-eigenvector of $A - 2I$. □

Problem 13. Let A be a $m \times n$ matrix.

(a) For each of $\operatorname{col}(A)$, $\operatorname{row}(A)$ and $\operatorname{null}(A)$, state which space $\mathbb{R}^{??}$ they are a subspace of.

(b) Why is $\dim \operatorname{row}(A) + \dim \operatorname{null}(A) = n$?

(c) Suppose that the columns of A are independent. What can you say about the dimensions of $\operatorname{col}(A)$, $\operatorname{row}(A)$ and $\operatorname{null}(A)$?

(d) Suppose that A has rank 2. What can you say about the dimensions of $\text{col}(A)$, $\text{row}(A)$ and $\text{null}(A)$?

Solution.

(a) $\text{col}(A)$ is a subspace of $\mathbb{R}^m$.

$\text{row}(A)$ is a subspace of $\mathbb{R}^n$.

$\text{null}(A)$ is a subspace of $\mathbb{R}^n$.

(b) $\dim \text{row}(A)$ is equal to the number of pivots, and $\dim \text{null}(A)$ equal to the number of free variables.

There is n columns, and each corresponds to a pivot or a free variable. Hence, $\dim \text{row}(A) + \dim \text{null}(A) = n$.

(c) $\dim \text{col}(A) = n$, $\dim \text{row}(A) = n$ and $\dim \text{null}(A) = 0$

(d) $\dim \text{col}(A) = 2$, $\dim \text{row}(A) = 2$ and $\dim \text{null}(A) = n - 2$ (because A has n columns, 2 of which contain a pivot). □

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