More on continued fractions

EG $[2;3,2,3,2,\dots] = x = 1+\sqrt{5/3}$

(a) Which number is represented by this CF?

$$x = 2 + \cfrac{1}{3+\cfrac{1}{\underbrace{2+\cfrac{1}{3+\dots}}_{=x}}}$$

$$x = 2 + \frac{1}{3+\frac{1}{x}} = 2 + \frac{x}{3x+1}$$

$$(x-2)(3x+1) = x$$

$$x = 1 \pm \sqrt{5/3} \approx \cancel{-0.291},\ 2.291$$

(b) Compute the first few convergents.

n	-2	-1	0	1	2	3
a_n			2	3	2	3
p_n	0	1	2	7	16	55
q_n	1	0	1	3	7	24

$c_0 = [2] = 2$

$c_1 = [2;3] = 2+\frac{1}{3}$

$$C_n = \frac{p_n}{q_n}$$

$c_2 = \frac{16}{7}$

$c_3 = \frac{55}{24}$

THM $[a_0; a_1, a_2, \dots]$ has convergents

$C_k = [a_0; a_1, \dots, a_k] = \frac{p_k}{q_k}$ characterized by:

$$p_k = a_k p_{k-1} + p_{k-2} \qquad q_k = a_k q_{k-1} + q_{k-2}$$

with $p_{-2}=0,\ p_{-1}=1$ $\qquad$ with $q_{-2}=1,\ q_{-1}=0$

$p_0 = a_0 p_{-1} + p_{-2} = a_0$ $\qquad$ $q_0 = a_0 q_{-1} + q_{-2} = 1$ $\qquad$ $C_0 = \frac{p_0}{q_0} = a_0$ ✓

$p_1 = a_1 p_0 + p_{-1} = a_0 a_1 + 1$ $\qquad$ $q_1 = a_1 q_0 + q_{-1} = a_1$ $\qquad$ $C_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1}$ ✓

$\vdots$

$$C_{n+1} = [a_0; a_1, \dots, a_n, a_{n+1}] = [a_0; a_1, \dots, \overbrace{a_n + \tfrac{1}{a_{n+1}}}^{\tilde{a}_n}]$$ ← shorter by 1

$$= \frac{\tilde{p}_n}{\tilde{q}_n} = \frac{\tilde{a}_n p_{n-1} + p_{n-2}}{\tilde{a}_n q_{n-1} + q_{n-2}} = \frac{(a_n a_{n+1}+1)p_{n-1} + a_{n+1}p_{n-2}}{(a_n a_{n+1}+1)q_{n-1} + a_{n+1}q_{n-2}}$$

$\tilde{a}_n = a_n + \frac{1}{a_{n+1}}$

$$= \frac{a_{n+1}\underbrace{(a_n p_{n-1} + p_{n-2})}_{p_n} + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} = \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}}$$