# Problems #2



**Problem 1. (warmup, 2 XP)** Suppose that the sequence $(a_n)_{n\geqslant 0}$ has ordinary generating function $F(x)$. For each of the following choices of $b_n$, express the generating function of $(b_n)_{n\geqslant 0}$ in terms of $F(x)$.

(a) $b_n = a_{n+3}$

(b) $b_n = n^2 a_n$

(c) $b_n = (n+1)(a_n - 2)$

(d) $b_n = a_{2n}$

(e) $b_n = \sum_{k=0}^{n} (-1)^k a_k$

**Solution.** Let us denote the generating function of $(b_n)_{n\geqslant 0}$ as $G(x)$.

(a) $G(x) = \frac{F(x) - a_0 - a_1 x - a_2 x^2}{x^3}$

(b) $G(x) = (xD)^2 F(x) = (x^2 D^2 + xD) F(x) = x^2 F''(x) + x F'(x)$

(c) $G(x) = (xD + 1)\left(F(x) - \frac{2}{1-x}\right) = xF'(x) + F(x) - \frac{2x}{(1-x)^2} - \frac{2}{1-x} = xF'(x) + F(x) - \frac{2}{(1-x)^2}$

(d) $G(x) = \frac{1}{2}(F(\sqrt{x}) + F(-\sqrt{x}))$

(e) The ogf of $((-1)^n a_n)_{n\geqslant 0}$ is $F(-x)$. Hence, the ogf of its partial sums is $G(x) = \frac{F(-x)}{1-x}$. □

**Problem 2. (warmup, 1 XP)** Show that $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$. [Use binomial and/or geometric series!]

**Solution.**

- Using the binomial series, we have

$$\frac{1}{(1-x)^{k+1}} = \sum_{n\geqslant 0} \binom{-k-1}{n} (-x)^n,$$

and so we need to show that

$$(-1)^n \binom{-k-1}{n} = \binom{n+k}{k},$$

which is true because

$$(-1)^n \binom{-k-1}{n} = (-1)^n \frac{(-k-1)(-k-2)\cdots(-k-n)}{n!} = \frac{(k+1)(k+2)\cdots(k+n)}{n!} = \binom{n+k}{n} = \binom{n+k}{k}.$$

- Alternatively, we can expand each geometric series to get

$$\frac{1}{(1-x)^{k+1}} = (1 + x + x^2 + x^3 + \ldots)^{k+1}.$$

The claim is that the coefficient of $x^n$ in that expression is $\binom{n+k}{k}$. Combinatorially, we have $k+1$ colors and $n$ indistinguishable objects which get colored. How many possible outcomes are there? The key idea is to think about placing the $n$ objects in a string, beginning with those in the first color, followed by a separator, then those of the second color, followed by a separator, and so on. In the end, we have $n+k$ things lined up ($n$ objects and $k+1-1$ separators). What matters is only the position of the $k$ separators. Since there are $n+k$ positions available, the total number of possibilities is $\binom{n+k}{k}$.

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- We can also start with the geometric series

$$\frac{1}{1-x} = \sum_{n \geqslant 0} x^n$$

and differentiate $k$ times to obtain

$$\frac{k!}{(1-x)^{k+1}} = \sum_{n \geqslant k} n(n-1)\cdots(n-k+1)x^{n-k} = \sum_{n \geqslant 0} (n+k)(n+k-1)\cdots(n+1)x^n,$$

and it only remains to divide both sides by $k!$.

- Here is a fourth proof, using the geometric series and an induction argument based on our knowledge of generating functions. The claim is true for $k=0$, because this is just the geometric series. Assume, for induction, that the claim is true for $k$. That is, $(1-x)^{-k-1}$ is the ogf of $\binom{n+k}{k}$. Then,

$$\frac{1}{(1-x)^{k+2}} = \frac{1}{1-x}\frac{1}{(1-x)^{k+1}}$$

is the ogf of the partial sums

$$\sum_{j=0}^{n} \binom{j+k}{k}.$$

It therefore suffices to establish

$$\sum_{j=0}^{n} \binom{j+k}{k} = \binom{n+k+1}{k+1},$$

which can be proved by using Pascal's basic relation

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

to make the sum telescope

$$\sum_{j=0}^{n} \binom{j+k}{k} = \sum_{j=0}^{n} \left(\binom{j+k+1}{k+1} - \binom{j+k}{k+1}\right) = \binom{n+k+1}{k+1}. \qquad \square$$

**Problem 3. (2 XP)** Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ be the harmonic numbers. Show that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$.

**Solution.**

- Recall that

$$\sum_{n=1}^{\infty} H_n x^n = \frac{1}{1-x}\log\left(\frac{1}{1-x}\right).$$

Hence, the ogf of the left-hand side is

$$\frac{1}{(1-x)^2}\log\left(\frac{1}{1-x}\right).$$

On the other hand, the ogf of the right-hand side is

$$\begin{aligned}\left((xD+1)\left[\frac{1}{1-x}\log\left(\frac{1}{1-x}\right)\right]\right) - (xD)\frac{1}{1-x} &= x\left(\frac{1}{(1-x)^2}\log\left(\frac{1}{1-x}\right) + \frac{1}{(1-x)^2}\right) + \frac{1}{1-x}\log\left(\frac{1}{1-x}\right) - \frac{x}{(1-x)^2} \\ &= \frac{1}{(1-x)^2}\log\left(\frac{1}{1-x}\right).\end{aligned}$$

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The generating functions match, and so the identity is true for all $n \geqslant 1$.

- Alternatively, we can give a direct derivation as follows:

$$\begin{aligned}\sum_{k=1}^{n} H_k &= n\cdot 1 + (n-1)\cdot\frac{1}{2} + (n-2)\cdot\frac{1}{3} + \ldots + (n-(n-1))\cdot\frac{1}{n}\\ &= nH_n - \left(\frac{1}{2} + \frac{2}{3} + \ldots + \frac{n-1}{n}\right)\\ &= nH_n - (n - H_n)\\ &= (n+1)H_n - n\end{aligned}$$

□

**Problem 4. (2 XP)** Suppose that the sequence $(a_n)_{n\geqslant 0}$ has ordinary generating function $F(x)$.

(a) Express the ordinary generating function for $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$ in terms of $F(x)$.

(b) The *binomial transform* of a sequence $a_n$ is the sequence $b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k$. What is the binomial transform of the binomial transform of a sequence?

**Solution.**

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} a_k x^n = \sum_{k=0}^{\infty} a_k \sum_{n=k}^{\infty}\binom{n}{k} x^n = \sum_{k=0}^{\infty} a_k \frac{x^k}{(1-x)^{k+1}}\\ &= \frac{1}{1-x} F\left(\frac{x}{1-x}\right)\end{aligned}$$

(b) The ogf of $((-1)^n a_n)_{n\geqslant 0}$ is $F(-x)$. Therefore, by the previous part, the binomial transform has generating function $G(x) = \frac{1}{1-x} F\left(\frac{x}{x-1}\right)$.

Hence, the binomial transform of the binomial transform has generating function

$$\frac{1}{1-x} G\left(\frac{x}{x-1}\right) = \frac{1}{1-x}\frac{1}{1-\frac{x}{x-1}} F\left(\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}\right) = F(x).$$

This means that the binomial transform of the binomial transform of $a_n$ is $a_n$ itself. In other words, the binomial transform is an involution. □

**Problem 5. (3 XP)** The exponential generating function of a sequence $(a_n)_{n\geqslant 0}$ is the (formal) power series $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Suppose that the sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ have exponential generating functions $F(x)$ and $G(x)$.

(a) Which sequence is generated by $F'(x)$? By $xF(x)$? By $F(x)G(x)$?

(b) What is the exponential generating function of $n a_n$? Of $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$?

(c) What is the exponential generating function of the binomial transform of $a_n$? Revisit the question what the binomial transform of the binomial transform of a sequence is.

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**Solution.**

(a) $F'(x) = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}$ generates the sequence $a_{n+1}$.

$xF(x) = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty} (n+1)a_n \frac{x^{n+1}}{(n+1)!}$ generates the sequence $na_{n-1}$.

$F(x)G(x) = \left(\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}\right) = \sum_{n,m=0}^{\infty} a_n b_m \frac{x^{n+m}}{n!m!} = \sum_{N=0}^{\infty} \sum_{k=0}^{N} a_k b_{n-k} \frac{x^N}{k!(N-k)!} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$

generates the sequence

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

(b) The exponential generating function of $na_n$ is $xF'(x)$.

It follows directly from the convolution formula above, and the fact that $e^x$ generates $(1)_{n \geqslant 0}$, that the exponential generating function of $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$ is $e^x F(x)$.

(c) The exponential generating function of the binomial transform of $a_n$ is $e^{-x}F(-x)$.

The binomial transform of the binomial transform of $a_n$ is $a_n$ itself, because if $G(x) = e^{-x}F(-x)$, then $e^{-x}G(-x) = e^{-x}(e^x F(x)) = F(x)$. □

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