Problems #3

Problem 1. (1 XP) Suppose that $(a_n)_{n\geqslant 0}$ has ogf $F(x)$. Which sequence is generated by $F(x)^k$, with $k\in\mathbb{Z}_{>0}$?

Solution. The sequence $(c_n)_{n\geqslant 0}$ generated by $F(x)^k$ is

$$c_n = \sum_{\substack{n_1\geqslant 0, n_2\geqslant 0, \ldots, n_k\geqslant 0\\ n_1+\ldots+n_k=n}} a_{n_1}\cdots a_{n_k}.$$

If this is to be used for practical purposes, we can optimize this a bit by noting that terms like $a_1a_1a_2$ and $a_1a_2a_1$ can be grouped together. This leads to

$$c_n = \sum_{\substack{m_0\geqslant 0, m_1\geqslant 0, \cdots, m_n\geqslant 0\\ m_0+m_1+\ldots+m_n=k\\ m_1+2m_2+\ldots+nm_n=n}} \frac{k!}{m_1!m_2!\cdots m_n!} a_0^{m_0}a_1^{m_1}\cdots a_n^{m_n}.$$ □

Problem 2. (1 XP) Let R be a ring. Which are the invertible elements in the ring $R[[x]]$ of formal power series?

Solution. A formal power series

$$a_0 + a_1x + a_2x^2 + \ldots$$

is invertible in $R[[x]]$ if and only if a_0 is invertible in R. Indeed, note that

$$\left(\sum_{n\geqslant 0} a_nx^n\right)\left(\sum_{n\geqslant 0} b_nx^n\right) = 1$$

is equivalent to $a_0b_0 = 1$ and, for all $n\geqslant 1$,

$$\sum_{k=0}^{n} a_kb_{n-k} = 0.$$

Hence, the invertibility of a_0 in R is a necessary condition. To see that it is also sufficient, observe that the convolution sum can be rewritten as

$$b_n = -\frac{1}{a_0}\sum_{k=1}^{n} a_kb_{n-k},$$

which allows us to define the coefficients b_n recursively. □

Problem 3. (2 XP)

(a) Give a generating function proof of the identity $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$.

(b) Also, show how the identity can be deduced from Binet's formula.

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Solution.

(a) Let $F(x) = x / (1 - x - x^2)$ be the generating function of the Fibonacci numbers F_n. It follows that

$$\sum_{n \geqslant 0} F_{2n} x^{2n} = \frac{F(x) + F(-x)}{2} = \frac{x^2}{1 - 3x^2 + x^4}, \quad \sum_{n \geqslant 0} F_{2n} x^n = \frac{x}{1 - 3x + x^2}.$$

Likewise,

$$\sum_{n \geqslant 0} F_{2n+1} x^{2n+1} = \frac{F(x) - F(-x)}{2} = \frac{x(1 - x^2)}{1 - 3x^2 + x^4}, \quad \sum_{n \geqslant 0} F_{2n+1} x^n = \frac{1 - x}{1 - 3x + x^2}.$$

The desired identity therefore translates into the generating function identity

$$\frac{1}{1 - x} \frac{x}{1 - 3x + x^2} = \frac{1 - x}{1 - 3x + x^2} - \frac{1}{1 - x},$$

which indeed holds true.

(b) Let $\varphi = \left(1 + \sqrt{5}\right) / 2$ and $\psi = \left(1 - \sqrt{5}\right) / 2$ be the roots of $x^2 - x - 1$. Starting by summing the geometric sum, we therefore have

$$\sum_{k=1}^{n} F_{2k} = \sum_{k=1}^{n} \frac{\varphi^{2k} - \psi^{2k}}{\varphi - \psi} = \frac{1}{\varphi - \psi} \left[\frac{\varphi^{2(n+1)} - 1}{\varphi^2 - 1} - \frac{\psi^{2(n+1)} - 1}{\psi^2 - 1} \right] = \frac{\varphi^{2n+1} - \psi^{2n+1}}{\varphi - \psi} - 1 = F_{2n+1} - 1.$$

Here, we used that $\varphi^2 - 1 = \varphi$ and $\psi^2 - 1 = \psi$, as well as $-1 / \varphi = \psi$. □

Problem 4. (2 XP) The Bessel differential equation is the second-order equation

$$x^2 y'' + x y' + (x^2 - \alpha^2) y = 0.$$

For simplicity, we will only consider the case $\alpha = 0$ here.

(a) Assume there is a power series solution $y(x) = \sum_{n \geqslant 0} a_n x^n$ (that is, a solution which is analytic at $x = 0$), normalized so that $a_0 = 1$. Translate the differential equation into a recurrence for the coefficients a_n.

(b) Solve that recurrence.

(c) Write down the corresponding solution of the differential equation. This is the Bessel function $J_0(x)$.

Solution.

(a) Substituting the power series into the differential equation, we obtain

$$x^2 y'' + x y' + (x^2 - \alpha^2) y = \sum_{n \geqslant 0} (n(n-1) a_n x^n + n a_n x^n + a_n x^{n+2}) = a_1 x + \sum_{n \geqslant 2} (n^2 a_n + a_{n-2}) x^n = 0.$$

Hence, $y(x) = \sum_{n \geqslant 0} a_n x^n$ is a solution if and only if $a_1 = 0$ and, for all $n \geqslant 2$,

$$n^2 a_n + a_{n-2} = 0.$$

(b) We conclude that $a_{2n+1} = 0$ for all $n \geqslant 0$. For the even indices, we have the recurrence

$$(2n)^2 a_{2n} + a_{2(n-1)} = 0,$$

that is,

$$a_{2n} = -\frac{1}{4n^2}\,a_{2(n-1)} = \left(-\frac{1}{4}\right)^2 \frac{1}{(n(n-1))^2} a_{2(n-2)} = \ldots = \left(-\frac{1}{4}\right)^n \frac{1}{(n!)^2} a_0 = \frac{(-1)^n}{(n!)^2 4^n}.$$

(c) We conclude that the corresponding solution of the differential equation is

$$y(x) = \sum_{n\geqslant 0} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n},$$

the Bessel function $J_0(x)$. □

Problem 5. (2 XP) Denote with B_n the Bernoulli numbers.

(a) Show that all, but the first, odd Bernoulli numbers are zero, that is, $B_{2n+1} = 0$ for all $n \geqslant 1$.

(b) Show that Euler's identity

$$\frac{1}{n}\sum_{k=1}^{n} \binom{n}{k} B_k B_{n-k} + B_{n-1} = -B_n$$

is true for all $n \geqslant 1$.

Solution.

(a) Let $F(x) = x/(e^x - 1)$ be the exponential generating function for the Bernoulli numbers. Then,

$$\sum_{n\geqslant 0} B_{2n+1}\frac{x^{2n+1}}{(2n+1)!} = \frac{F(x) - F(-x)}{2} = -\frac{x}{2},$$

which shows that $B_{2n+1} = 0$ with the only exception for $n = 0$, in which case $B_1 = -1/2$.

(b) First, let us rewrite the identity as

$$\sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} - B_n + n B_{n-1} = -n B_n. \tag{1}$$

The exponential generating function of this is

$$\left(\frac{x}{e^x - 1}\right)^2 - \frac{x}{e^x - 1} + x\,\frac{x}{e^x - 1} = -x D\,\frac{x}{e^x - 1}.$$

Simplifying both sides, we obtain

$$\frac{x^2 + (x^2 - x)(e^x - 1)}{(e^x - 1)^2} = -x\,\frac{(e^x - 1) - x e^x}{(e^x - 1)^2},$$

and these are indeed equal, so that (1) is actually true for all $n \geqslant 0$. □

Problem 6. (2 XP)

(a) Take the logarithm of both sides of Euler's product formula and differentiate to prove that

$$n p(n) = \sum_{k=0}^{n-1} p(k)\sigma(n-k),$$

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where $\sigma(n)$ is the sum of the divisors of n.

(b) Let $p(n,k)$ be the number of partitions of n into k parts. Generalize Euler's product formula to the bivariate generating function

$$\sum_{n,k\geqslant 0} p(n,k)x^n y^k.$$

Solution.

(a) Recall that Euler's identity states

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n\geqslant 1} \frac{1}{1-x^n}.$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}\log F(x) = \frac{F'(x)}{F(x)}$, taking the logarithm of both sides and differentiating yields

$$\sum_{n=1}^{\infty} np(n)x^{n-1} \Big/ \sum_{n=0}^{\infty} p(n)x^n = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n\geqslant 1} \log\left(\frac{1}{1-x^n}\right) = \sum_{n\geqslant 1} \frac{nx^{n-1}}{1-x^n},$$

which we rearrange to

$$\sum_{n=1}^{\infty} np(n)x^n = \left(\sum_{n=0}^{\infty} p(n)x^n\right)\left(\sum_{n\geqslant 1} \frac{nx^n}{1-x^n}\right).$$

The claim now follows from comparing coefficients and noting that

$$\sum_{n\geqslant 1} \frac{nx^n}{1-x^n} = \sum_{n\geqslant 1} n(x^n + x^{2n} + x^{3n} + \ldots) = \sum_{m=1}^{\infty} \sigma(m)x^m.$$

See also: http://mathoverflow.net/questions/127000/partitions-sum-of-divisors-identity

(b) We can proceed along the lines of our derivation of Euler's product identity. Recall that we interpreted the factor

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + \ldots$$

as describing how often the part n occurs in a partition. Replacing each such factor with

$$1 + yx^n + y^2x^{2n} + \ldots = \frac{1}{1-yx^n},$$

to keep track of the number of parts as the exponent of y, we arrive at

$$\sum_{n,k\geqslant 0} p(n,k)x^n y^k = \prod_{n\geqslant 1} \frac{1}{1-yx^n}.$$

We refer to [Wilf, *Generatingfunctionology*, p. 100] for a systematic generalization of generating functions of this kind. □

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