Problems #3

Special Functions & WZ Theory
AARMS Summer School, Dalhousie University
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Problem 1. (1 XP) Suppose that $(a_n)_{n \geqslant 0}$ has ogf $F(x)$. Which sequence is generated by $F(x)^k$, with $k \in \mathbb{Z}_{>0}$?

Problem 2. (1 XP) Let R be a ring. Which are the invertible elements in the ring $R[[x]]$ of formal power series?

Problem 3. (2 XP)

(a) Give a generating function proof of the identity $\displaystyle\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$.

(b) Also, show how the identity can be deduced from Binet's formula.

Problem 4. (2 XP) The Bessel differential equation is the second-order equation

$$x^2 y'' + x y' + (x^2 - \alpha^2) y = 0.$$

For simplicity, we will only consider the case $\alpha = 0$ here.

(a) Assume there is a power series solution $y(x) = \sum_{n \geqslant 0} a_n x^n$ (that is, a solution which is analytic at $x = 0$), normalized so that $a_0 = 1$. Translate the differential equation into a recurrence for the coefficients a_n.

(b) Solve that recurrence.

(c) Write down the corresponding solution of the differential equation. This is the Bessel function $J_0(x)$.

Problem 5. (2 XP) Denote with B_n the Bernoulli numbers.

(a) Show that all, but the first, odd Bernoulli numbers are zero, that is, $B_{2n+1} = 0$ for all $n \geqslant 1$.

(b) Show that Euler's identity

$$\frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} B_k B_{n-k} + B_{n-1} = -B_n$$

is true for all $n \geqslant 1$.

Problem 6. (2 XP)

(a) Take the logarithm of both sides of Euler's product formula and differentiate to prove that

$$n p(n) = \sum_{k=0}^{n-1} p(k) \sigma(n-k),$$

where $\sigma(n)$ is the sum of the divisors of n.

(b) Let $p(n, k)$ be the number of partitions of n into k parts. Generalize Euler's product formula to the bivariate generating function

$$\sum_{n, k \geqslant 0} p(n, k) x^n y^k.$$

Armin Straub
straub@southalabama.edu