Problems #6

Special Functions & WZ Theory
AARMS Summer School, Dalhousie University
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Problem 1. Let $D = \frac{\mathrm{d}}{\mathrm{d}x}$.

(a) **(1 XP)** What is the commutator of D and x^n? In other words, compute $Dx^n - x^n D$.

(b) **(1 XP)** Write $(xD)^3$ as a linear combination of terms of the form $x^a D^b$.

(c) **(2 XP extra)** Come up with (and, possibly prove) a general formula for $(xD)^n$ in the spirit above.

You are encouraged to let Sage help you get an idea. For instance, check out how to use `FreeAlgebra` to create a non-commutative free algebra `A`, and then check out `A.g_algebra` for declaring commutation relations.

If this tickles your fancy, this could be turned into a Sage project.

Solution.

(a) $Dx^n - x^n D = n x^{n-1}$

(b) We just observed that $Dx^n = x^n D + n x^{n-1}$. Using that a few times, we obtain

$$\begin{aligned}
(xD)^2 &= x(Dx)D = x(xD+1)D = x^2D^2 + xD \\
(xD)^3 &= xD(xD)^2 = xD(x^2D^2 + xD) = x(Dx^2)D^2 + (xD)^2 = x(x^2D + 2x)D^2 + x^2D^2 + xD \\
&= x^3D^3 + 3x^2D^2 + xD
\end{aligned}$$

(c) Let's have Sage do all the heavy lifting:

```
Sage] F.<x,Dx> = FreeAlgebra(QQ,2)
Sage] U = F.g_algebra({Dx*x: x*Dx+1})
Sage] U.inject_variables()
   Defining x, Dx
Sage] (x*Dx)^3
```

$x^3Dx^3 + 3x^2Dx^2 + xDx$

```
Sage] (x*Dx)^5
```

$x^5Dx^5 + 10x^4Dx^4 + 25x^3Dx^3 + 15x^2Dx^2 + xDx$

Conjecture away! □

Problem 2. (1 XP) Determine the ordinary generating function of the squares F_n^2 of the Fibonacci numbers.

Solution. We showed in class that $F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$. Hence, proceeding as we did for the Fibonacci numbers themselves, we derive that

$$\sum_{n=0}^{\infty} F_n^2 x^n = \frac{x - x^2}{1 - 2x - 2x^2 + x^3} = \frac{x(1-x)}{(1+x)(1-3x+x^2)}.$$

Here is a reminder how we obtained the recursion for F_n^2. The Fibonacci numbers are annihilated by the operator

$$S^2 - S - 1 = (S - \varphi)(S - \psi),$$

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where $\varphi = \left(1+\sqrt{5}\right)/2$ and $\psi = \left(1-\sqrt{5}\right)/2$. Consequently, F_n can be expressed as a linear combination of φ^n and ψ^n. It follows that F_n^2 can be expressed as a linear combination of φ^{2n}, ψ^{2n} and $(\varphi\psi)^n$. In particular, these numbers are annihilated by

$$(S-\varphi^2)(S-\psi^2)(S-\varphi\psi) = (S^2-3S+1)(S+1) = S^3-2S^2-2S+1.$$

In other words, we have $F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0$. □

Problem 3. (1 XP) Let $d > 0$ be an integer. Prove that there are constants α, β such that, for all $n \geqslant 0$,

$$F_{n+d} = \alpha F_n + \beta F_{n+1}.$$

Solution. The sequences $(F_n)_{n\geqslant 0}$, $(F_{n+1})_{n\geqslant 0}$ and $(F_{n+d})_{n\geqslant 0}$ are all elements of the two dimensional space of solutions to the recurrence

$$(S^2-S-1)X_n = 0.$$

Since $(F_n)_{n\geqslant 0}$ and $(F_{n+1})_{n\geqslant 0}$ are linearly independent (they are clearly not just multiples of each other), they form a basis for that space. In particular, $(F_{n+d})_{n\geqslant 0}$ can be written as a (unique) linear combination of these two sequences. □

Problem 4. (1 XP) Why is it impossible for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ to satisfy a linear recurrence with constant coefficients?

Solution. We showed in class that the ordinary generating function for the Catalan numbers is

$$\sum_{n\geqslant 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Since their generating function is not rational, the Catalan numbers are not C-finite. □

Problem 5. The purpose of this problem is to look at linear differential equations with constant coefficients, and to observe how transparent the theory becomes when viewing them through our operator glasses. So, put on those glasses!

(a) **(1 XP)** What is the general solution to the differential equation $y'' - y' - 6y = 0$?

(b) **(1 XP)** What is the general solution to the differential equation $y'' - 6y' + 9y = 0$?

(c) **(1 XP)** Come up with a theorem that provides a basis for the solutions to any homogeneous linear differential equation

$$y^{(k)} + c_{k-1}y^{(k-1)} + \ldots + c_1 y' + c_0 y = 0.$$

(d) **(1 XP)** What is the general solution to the differential equation $y'' - y' - 6y = e^x$?

Hint: Can you reduce to the homogeneous case?

(e) **(1 XP)** What is the general solution to the differential equation $y'' - y' - 6y = 2e^{3x}$?

Solution.

(a) In operator notation, this differential equation is

$$(D^2 - D - 6)y = 0,$$

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which we can factor as $(D-3)(D+2)y=0$. In this form, it is obvious that we have the two special solutions $y(x)=e^{3x}$ and $y(x)=e^{-2x}$. On the other hand, the space of solutions is a two dimensional vector space, and so we conclude that the general solution is

$$y(x)=c_1 e^{3x}+c_2 e^{-2x}.$$

(b) We proceed as before, and write the differential equation as

$$(D^2-6D+9)y=(D-3)^2 y=0.$$

One solution is again obvious: $y(x)=e^{3x}$. We find a second independent solution by solving the inhomogeneous differential equation

$$(D-3)y=e^{3x}$$

of order 1 (this is the same approach we took to find missing solutions in the case of recurrences with constant coefficients). Variation of constants (or an educated guess) leads to $y(x)=x e^{3x}$. For our original differential equation, we have therefore found the general solution

$$y(x)=(c_0+c_1 x)e^{3x}.$$

(c) Let $r_1,r_2,\ldots,r_d$ be the distinct roots of $p(D)=D^k+c_{k-1}D^{k-1}+\ldots+c_0$, and denote with $m_1,m_2,\ldots,m_d$ their multiplicity. Then a basis for the differential equation $p(D)y=0$ is given by the functions

$$\{x^k e^{r_j} : j\in\{1,2,\ldots d\}, k\in\{0,1,\ldots,m_j-1\}\}.$$

(d) We write $y''-y'-6y=e^x$ as

$$(D-3)(D+2)y=e^x,$$

and multiply both sides with $D-1$ to obtain the homogeneous differential equation

$$(D-1)(D-3)(D+2)y=0.$$

Hence, any solution to our differential equation must be of the form

$$y(x)=c_1 e^x+c_2 e^{3x}+c_3 e^{-2x}.$$

In fact, disregarding solutions to the associated homogeneous equation, we know there must a solution of the form ce^x. We determine the value of c by substituting into the differential equation:

$$(D-3)(D+2)ce^x=c(1-3)(1+2)e^x=-6ce^x\overset{!}{=}e^x$$

Hence, we have found the particular solution $y(x)=-\frac{1}{6}e^x$. The general solution is

$$y(x)=-\frac{1}{6}e^x+c_1 e^{3x}+c_2 e^{-2x}.$$

(e) Homogenizing as before, this time multiplying with $D-3$, we obtain the differential equation

$$(D-1)(D-3)^2 y=0,$$

which shows that any solution to our original differential equation must be of the form

$$y(x)=c_1 e^x+(c_2+c_3 x)e^{3x}.$$

Again, this implies that there must a solution of the form cxe^{3x}.

$$(D+2)(D-3)cxe^{3x}=c(D+2)e^{3x}=5ce^{3x}\overset{!}{=}2e^{3x}$$

Hence, the general solution to our original differential equation is

$$y(x)=c_1 e^x+\left(c_2+\frac{2}{5}x\right)e^{3x}.$$ □