Problems #6

Special Functions & WZ Theory

AARMS Summer School, Dalhousie University
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Problem 1. Let $D = \frac{\mathrm{d}}{\mathrm{d}x}$.

(a) **(1 XP)** What is the commutator of D and x^n? In other words, compute $Dx^n - x^n D$.

(b) **(1 XP)** Write $(xD)^3$ as a linear combination of terms of the form $x^a D^b$.

(c) **(2 XP extra)** Come up with (and, possibly prove) a general formula for $(xD)^n$ in the spirit above.

You are encouraged to let Sage help you get an idea. For instance, check out how to use `FreeAlgebra` to create a non-commutative free algebra `A`, and then check out `A.g_algebra` for declaring commutation relations.

If this tickles your fancy, this could be turned into a Sage project.

Problem 2. **(1 XP)** Determine the ordinary generating function of the squares F_n^2 of the Fibonacci numbers.

Problem 3. **(1 XP)** Let $d > 0$ be an integer. Prove that there are constants α, β such that, for all $n \geqslant 0$,

$$F_{n+d} = \alpha F_n + \beta F_{n+1}.$$

Problem 4. **(1 XP)** Why is it impossible for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ to satisfy a linear recurrence with constant coefficients?

Problem 5. The purpose of this problem is to look at linear differential equations with constant coefficients, and to observe how transparent the theory becomes when viewing them through our operator glasses. So, put on those glasses!

(a) **(1 XP)** What is the general solution to the differential equation $y'' - y' - 6y = 0$?

(b) **(1 XP)** What is the general solution to the differential equation $y'' - 6y' + 9y = 0$?

(c) **(1 XP)** Come up with a theorem that provides a basis for the solutions to any homogeneous linear differential equation

$$y^{(k)} + c_{k-1} y^{(k-1)} + \ldots + c_1 y' + c_0 y = 0.$$

(d) **(1 XP)** What is the general solution to the differential equation $y'' - y' - 6y = e^x$?

Hint: Can you reduce to the homogeneous case?

(e) **(1 XP)** What is the general solution to the differential equation $y'' - y' - 6y = 2e^{3x}$?

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