Problems #7

Special Functions & WZ Theory

AARMS Summer School, Dalhousie University
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Problem 1. (2 XP) Let $\mathcal{X}$ be the vector space of solutions to the differential equation

$$y^{(d)} + c_{d-1} y^{(d-1)} + \ldots + c_1 y' + c_0 y = 0,$$

and let $\mathcal{Y}$ be the vector space of solutions to the recurrence

$$a_{n+d} + c_{d-1} a_{n+d-1} + \ldots + c_1 a_{n+1} + c_0 a_n = 0.$$

Show that the map EGF: $\mathcal{Y} \rightarrow \mathcal{X}$ defined by $(a_n)_{n \geqslant 0} \mapsto \sum_{n \geqslant 0} a_n \frac{x^n}{n!}$ is an isomorphism.

Problem 2. (1 XP) True or false? Any eventually periodic sequence is C-finite.

Problem 3. (2 XP) The Chebyshev polynomials $T_n(x)$ of the first kind are the unique polynomials satisfying

$$T_n(\cos\theta) = \cos(n\theta).$$

Prove that the sequence $(T_n(x))_{n \geqslant 0}$ is C-finite.

Problem 4. (3 XP) Recall that the Bernoulli polynomials $B_n(t)$ are the polynomials characterized by

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{x e^{t x}}{e^x - 1}.$$

(a) Show that the Bernoulli polynomials satisfy $B_n'(t) = n B_{n-1}(t)$.

(b) Further, show that, for $n \geqslant 1$, the Bernoulli polynomials satisfy $\int_0^1 B_n(t)\,\mathrm{d}t = 0$.

(c) Observe that the Bernoulli polynomials are characterized by the initial condition $B_0(t) = 1$ together with the two properties you just showed. Compute the first few Bernoulli polynomials via that route.

(d) Forget that you know the exponential generating function of the Bernoulli polynomials. *Derive* this generating function from the two properties above.

Problem 5. (1 XP) Show that the Bernoulli polynomials have the expansion $B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} t^k$.

Problem 6. (1 XP) Give a (rough) asymptotic estimate for the Bernoulli numbers B_{2n} as $n \rightarrow \infty$.

Problem 7. (2 XP) Let $B_n(x)$ denote the Bernoulli polynomials.

(a) Prove that $1^p + 2^p + \ldots + N^p = \frac{B_{p+1}(N+1) - B_{p+1}(1)}{p+1}$.

(b) Show that $1^3 + 2^3 + \ldots + N^3 = (1 + 2 + \ldots + N)^2$.

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