Problems #9

Special Functions & WZ Theory

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Note. If one of these problems appeals to you particularly, you could use it as a starting point for your Sage project.

Problem 1. Fix some $N \in \mathbb{Z}_{>0}$. In class, we showed that

$$\zeta(3) = \sum_{n=1}^{a-1} \frac{1}{n^3} + \frac{1}{2} \sum_{n=0}^{N} \frac{(-1)^n (n+1) B_n}{a^{n+2}} + O\left(\frac{1}{a^{N+1}}\right),$$

as $a \to \infty$.

(a) **(1 XP)** Euler used $a = 10$ to accurately compute $\zeta(3)$ to 17 decimal digits. Using Sage, compute approximations for several small values of N until you obtain, like Euler, at least 17 digits accuracy.

(b) **(1 XP)** Continue increasing N and observe that, eventually, the approximations get worse until they turn into complete rubbish. Make a plot of the number of correct digits versus N.

(c) **(2 XP)** Obtain a similar asymptotic expansion for $\zeta(s)$, with $s > 1$.

Problem 2. (up to 3 XP extra) Define the numbers C_n by the recurrence $C_n = C_{n-2} + C_{n-3}$ subject to the initial conditions $C_0 = 3$, $C_1 = 0$, $C_2 = 2$. Someone suggests that n is prime if and only if n divides C_n.

(a) Check this claim for small values of n.

(b) Investigate!

Problem 3. (up to 3 XP extra) Recall that the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of ways, in which a product of $n+1$ terms can be interpreted. For instance, the product $abcd$ can be interpreted in $C_3 = 5$ different ways:

$$((ab)c)d, \quad (a(bc))d, \quad (ab)(cd), \quad a((bc)d), \quad a(b(cd))$$

(a) **(for experts)** Write a function in Sage, which, given a list of terms, produces all these possible products. For flexibility, your function should also allow to make it possible to specify how two elements are to be multiplied.

(b) **(for anybody interested)** I have implemented such a function in the notebook `challenge-catalan`, which you can download from our website or find in our shared SageMathCloud project (copy it to your own project before working with it). Try to understand how it works. Look at the examples and comments.

(c) Say, we have chosen a multiplication and some way to generate lists of $n+1$ terms that we want to multiply. Let D_n be the number of distinct products that are thus obtained. We know that $D_n \leqslant C_n$, but, of course, there could be less distinct products. For instance, if multiplication is associative, then $D_n = 1$.

Can you find interesting instances in which D_n lies between these extremes?

Just as a suggestion, consider, for instance, the cross-product of vectors in $\mathbb{R}^3$.

Problem 4. (up to 3 XP extra) Fix a positive integer m. Investigate what happens when we consider the Fibonacci numbers modulo m.

(a) Do we always get a periodic sequence?

(b) Can you make a general statement for all C-finite sequences?

(c) Explore if you can say anything about the period in the periodic cases.

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